The total force on the planet is given by

\[ F = -\left(\frac{k}{r^2} + mcr\right)r \]

We want to calculate the period for a circular orbit of radius \( r_0 \).

The condition for a circular orbit is given by:

\[ \frac{dv}{dr} = 0 \quad \text{at} \quad r = r_0, \] where \( V(r) \) is the effective potential

\[ V(r) = V(r) + \frac{l^2}{2mr^2} \]

Then,

\[ \frac{dv}{dr} = \frac{dv}{dr} - \frac{l^2}{mr^3} \]

\[ = \left(\frac{k}{r^2} + mcr\right) - \frac{l^2}{mr^2} \]

\[ , \quad F = -\frac{dv}{dr} \]

Setting \( \frac{dv}{dr} = 0 \) at \( r = r_0 \), we have

\[ \frac{k}{r_0^2} + mcr_0 = \frac{l^2}{mr_0^2} \]

\[ kmr_0 + m^2c r_0^4 = l^2 \]

Now, recall that for a central force potential, \( l \) is a constant and is given by

\[ l = mr^2\dot{\theta} \]
And, we calculate the period of the circular orbit from the orbit's constant angular velocity \( \omega = \dot{\theta} \):

\[
T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi m r_0^3}{\lambda}
\]

Substitute what we have calculated for \( \lambda \), we have

\[
T = \frac{2\pi m r_0^2}{\sqrt{k m r_0 + m^2 c^2 r_0^4}}
\]

\[
T = \frac{2\pi}{\sqrt{\frac{k}{m r_0^3} + c}} = \frac{2\pi}{\omega_0}
\]

where \( \omega_0 = \sqrt{\frac{k}{m r_0^3} + c} \) is the angular velocity for the circular orbit.
Since the mass is constrained to move on the cone, 
\[ \tan \alpha = \frac{r}{z} \]

or 
\[ g(r, z, \theta) = z \tan \alpha - r = 0 \]

a). 
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \]

\[ T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) \] (improper) in coordinates \((r, \theta, z)\)

\[ T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \cot^2 \alpha \dot{r}^2) \] (proper) in coordinates \((r, \theta)\)

\[ V = mgz \] in improper coordinates

\[ V = mg \cot \alpha \] in proper coordinates

\[ L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \cot^2 \alpha \dot{r}^2) - m g \cot \alpha r \] (in proper generalized coordinates)

\[ L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - m g z \] (in improper generalized coordinates)
let calculate the equation of motion using both the proper and the improper generalized coordinates:

1. Proper \((r, \theta)\):

\[
\frac{\partial L}{\partial \dot{r}} = m \dot{r} + m \cot^2 \alpha \dot{r} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m(1 + \cot^2 \alpha) \dot{r} \quad = m \csc^2 \alpha \dot{r}
\]

\[
\frac{\partial L}{\partial r} = m \dot{r} \dot{\theta}^2 - mg \cot \alpha
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \implies \csc^2 \alpha \ddot{r} - \dot{r} \dot{\theta}^2 + g \cot \alpha = 0
\]

\[
\ddot{r} = r \dot{\theta}^2 \sin \alpha + g \cos \alpha \sin \alpha = 0 \tag{\star}
\]

2. \(\theta\):

\[
\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \quad \frac{\partial L}{\partial \theta} = 0
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \implies \frac{d}{dt} (mr^2 \dot{\theta}) = 0
\]

\[
\implies mr^2 \dot{\theta} = C \quad \text{angular momentum is conserved.}
\]

\(\text{(We will use this fact to reduce this problem to an effective one-dimensional problem.)}\)
\[ (2) \text{ Improper } (r, \theta, z): \]

Since we have an equation of constraint
\[ g(r, \theta, z) = z + \tan \alpha - r = 0, \]
we have one Lagrang multiplier \( \lambda \).

\[ \frac{\mathbf{r}}{\dot{r}} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m \ddot{r}, \quad \frac{\partial L}{\partial r} = m r \dot{\theta}^2, \quad \frac{\partial L}{\partial \theta} = -1 \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \lambda \frac{\partial g}{\partial r} \Rightarrow m \dddot{r} - m r \ddot{\theta}^2 = -\lambda \quad (1) \]

\[ \frac{\partial L}{\partial \theta} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta} \Rightarrow \frac{d}{dt} \left( m r^2 \ddot{\theta} \right) = 0 \]

or \( m r^2 \ddot{\theta} = \lambda \) (same as before)

\[ \frac{\partial L}{\partial \dot{\theta}} = m \ddot{\theta}, \quad \frac{\partial L}{\partial \dot{z}} = -mg \quad \frac{\partial L}{\partial z} = \tan \alpha \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \dot{z}} = \lambda \frac{\partial g}{\partial \dot{z}} \Rightarrow m \dddot{\theta} + mg = \lambda \tan \alpha \quad (2) \]

Now, we want to put in the equation of constraint, \( \cot \alpha \ddot{r} = \ddot{z} \)

plug this into (2): \( mg \cot \alpha \ddot{r} + mg = \lambda \tan \alpha \)

\[ m \cot^2 \alpha \dddot{r} + mg \cot \alpha = \lambda \]
Substitute $\lambda$ into (1), we have

$$m\ddot{r} - m\dot{r}^2 \dot{\theta}^2 = -mcot^2\alpha \dot{r} - mg\cot\alpha$$

$$\ddot{r} - r\dot{\theta}^2 + cot^2\alpha \dot{r} + mg \cot \alpha = 0$$

$$\csc^2 \alpha \ddot{r} - r \dot{\theta}^2 + mg \cot \alpha = 0$$

$$\ddot{r} = r \dot{\theta}^2 \sin^2\alpha + g \cos \alpha \sin \alpha$$

(same as Eq. 8 from Method 1)

So, to analyze this system, we have the following equation of motion

$$m\ddot{r} - m\dot{r}^2 \dot{\theta}^2 = -mg\cos \alpha \sin \alpha = 0$$

4) The angular momentum conservation equation

$$mr^2 \dot{\theta} = \ell$$

Using $\ell$, we can reduce this to a effective 1-D system:

$$m\ddot{r} - \frac{\ell^2}{mr^3} \sin^2 \alpha + mg \cos \alpha \sin \alpha = 0$$
\[ m \ddot{r} = -\frac{d}{dr} \left( \frac{L^2}{2mr^2} \sin^2 \alpha + mg \cos \alpha \sin \alpha \right) \]

\[ \ddot{r} = -\frac{dV}{dr} = f(r) \]

Where
\[ V(r) = \frac{L^2}{2mr^2} \sin^2 \alpha + mg \cos \alpha \sin \alpha \]
\[ = \left( \frac{L^2}{2mr^2} + mg \cot \alpha \right) \sin^2 \alpha \]

is the effective potential for the \( \beta \) motion.

and \( f(r) = -\frac{dV}{dr} \) is the effective force.

For this effective one-dimensional motion, we can qualitatively consider the behavior of the orbits using the following effective potential:

\[ \hat{V}(r) = \frac{V(r)}{\sin^2 \alpha} = \frac{L^2}{2mr^2} + mg \cot \alpha \]

\[ \hat{V}(r) \]

energy scale
rescaled by \( \sin^2 \alpha \)

\[ V(r) \]

\[ \frac{L^2}{2mr^2} \]

\[ r \]
Since $L$ does not depend on $t$ explicitly and $V(r)$ does not depend on $r$, $E = T + V$ is conserved as expected.

\[ E = \frac{1}{2} m c s^2 \alpha \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + m g r \cot \alpha \]

\[ E = \frac{1}{2} m c s^2 \alpha \dot{r}^2 + \frac{\ell^2}{2 m r^2} + m g r \cot \alpha \]

\[ \frac{1}{2} m \dot{r}^2 = E - \tilde{V}(r) \quad \text{where} \quad \hat{E} = E \sin^2 \alpha \quad \text{in units of energy unit} \]

\[ \tilde{V}(r) = \frac{\ell^2}{2 m r^2} + m g r \cot \alpha \]

- For $\hat{E} < \hat{E}_0$ (see graph above), similar to central force problems with $E$ & $L$ being constant, motion is not allowed since $KE(\dot{r})$ can't be negative.

- For $\hat{E} = \hat{E}_0$, the motion is a circular orbit at $r = r_0$.

We can solve for $r_0$ by setting

\[ \frac{d\tilde{V}}{dr} = 0 = - \frac{\ell^2}{m r^3} + m g \cot \alpha = 0 \]

\[ r_0 = \left( \frac{\ell^2}{m^2 g \cot \alpha} \right)^{\frac{1}{2}} \]
\[ E_0 = V(r_0) = \frac{e^2}{2mr^2} + m_\text{g} r \cot \alpha \]  

- For \( E > E_0 \), the orbit will be bounded between two radii \( (r_1, r_2) \). These two radii can be found by solving the following cubic equation:

\[
\frac{e^2}{2mr^2} + m_\text{g} r \cot \alpha = E \quad \text{intersection between } V(r) \text{ & } E.
\]

\[
e^2 + 2m^2 g r^3 \cot \alpha = 2mr^2 \]

\[(2m^2 g \cot \alpha) r^3 - 2mE r^2 + e^2 = 0\]

- Graphically, \( r_1 \) & \( r_2 \) are the two intersection points between \( V(r) \) & \( E \) for \( E > E_0 \).

motion of the particle will be bounded by the the planes with radii \( r_1 \) & \( r_2 \).
The cubic equation
\[ h(r) = (2m^2 \beta \gamma \cos \alpha) r^3 - 2m \beta r^2 + \beta^2 = 0 \]
will in general have three solutions.

- Two will correspond to \( r_1 \) and \( r_2 \) when \( \beta > \beta_0 \)
  and the third one is for a negative \( r \) (non-physical).

For \( \beta = \beta_0 \), the cubic will be a double root on the \( +x \) axis.

- For \( \beta < \beta_0 \), there will be no solutions on the \( +x \) axis.