3.13 For a central force problem with a power law potential, the equation of the orbit is given by Eq. 3.33:

\[ \frac{\ell}{r^2} \frac{d}{d\theta} \left( \frac{\ell}{mr^2} \frac{dr}{d\theta} \right) - \frac{\ell^2}{mr^2} = f(r) \]

where \( f(r) = -\frac{dv}{dr} = ar^n \)

a) We need to show that \( n = -5 \) for the following orbit:

From this figure, we can write down the following orbit equation:

\[ r = 2a \cos \theta \]

\( \ell \) can be simplify by multiply \( r^2 \) through out the equation:

\[ \ell \frac{d}{d\theta} \left( \frac{\ell}{mr^2} \frac{dr}{d\theta} \right) - \frac{\ell^2}{mr} = r^2 f(r) \]

\[ \frac{\ell}{mr^2} \frac{dr}{d\theta} = \frac{\ell}{m4a^2 \cos^2 \theta} (-2a \sin \theta) = -\frac{\ell}{2am} \frac{\sin \theta}{\cos^3 \theta} \]

\[ \ell \frac{d}{d\theta} \left( \frac{\ell}{mr^2} \frac{dr}{d\theta} \right) = -\frac{\ell^2}{2am} \frac{d}{d\theta} \left( \frac{\sin \theta}{\cos^3 \theta} \right) \]

\[ = -\frac{\ell^2}{2am} \left( \frac{\cos \theta}{\cos^3 \theta} + \left( \frac{-2\sin \theta}{\cos^3 \theta} \right) (-\sin \theta) \right) \]
\[ \ell^2 = -\frac{\ell^2}{2am} \left( \frac{1}{\cos \theta} + \frac{2 \sin^2 \theta}{\cos^3 \theta} \right) \]

\[ -\frac{\ell^2}{2am} \left( \frac{1}{\cos \theta} + \frac{2 \sin^2 \theta}{\cos^3 \theta} \right) - \frac{\ell^2}{2am \cos \theta} = 4a^2 \cos \theta \ f(r) \]

\[ -\frac{\ell^2}{2am} \left( \frac{\cos^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta}{\cos^2 \theta} \right) = 4a^2 \cos \theta \ f(r) \]

\[ f(r) = -\frac{\ell^2 a^2}{m} - \frac{8}{8 \cdot 4a^5 \cos^5 \theta} \]

\[ f(r) = -\frac{8\ell^2 a^2}{mr^5} \]

So, the force law is an inverse fifth power, i.e., \( n = -5 \).

5). Show that the total energy of the particle is zero.

\[ r = 2a \cos \theta \]
\[ \dot{r} = -2a \sin \theta \dot{\theta} \]
\[ T = \frac{1}{2} mr^2 + \frac{1}{2} mr^2 \dot{\theta}^2 \]
\[ = \frac{1}{2} m \left( 4a^2 \sin^2 \theta \dot{\theta}^2 + 4a^2 \cos^2 \theta \dot{\theta}^2 \right) \]
\[ T = 2ma^2 \dot{\theta}^2 \]

- For a central force problem, the angular momentum is a conserved quantity. We can use it to rewrite the above expression:

\[ l = mr^2 \dot{\theta}^2 \Rightarrow \ell^2 = m^2 r^4 \dot{\theta}^4 \]

So, \[ T = 2ma^2 \left( \frac{\ell^2}{m^2 r^4} \right) \]

\[ = \frac{2 \ell^2 a^2}{m r^4} \]

- \[ f(r) = -\frac{8 \ell^2 a^2}{mr^5} \Rightarrow V(r) = -\int f(r) \, dr \]

\[ = -\frac{2 \ell^2 a^2}{mr^4} \]

- Now, \[ E = T + V \]

\[ = \frac{2 \ell^2 a^2}{m r^4} - \frac{2 \ell^2 a^2}{m r^4} = 0 \]

c). Find the period of motion:

Going back to the orbit picture, the particle goes around one cycle when \( \theta \) goes from \(-\pi/2\) to \( \pi/2\).
\[ T = \int_{0}^{\pi/2} dt = \int_{0}^{\pi/2} \frac{d\theta}{\sin \theta} \]

\[ = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin \theta} = \int_{-\pi/2}^{\pi/2} \frac{mr^2}{l} \, d\theta \quad \text{(again, we use the relation)} \]

\[ = \frac{4a^2m}{l} \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \left( 1 + \frac{\cos \theta}{4 \sin^2 \theta} \right) \, d\theta \]

\[ = \frac{4a^2m}{l} \left( \frac{\pi}{4} + \frac{1}{4} \sin 2\theta \right)_{-\pi/2}^{\pi/2} \]

\[ = \frac{4a^2m}{l} \left( \frac{\pi}{4} + \frac{1}{4} (\sin \pi - \sin (-\pi)) \right) = \frac{2\pi ma^2}{l} \]

---

d). Choosing the force center as the origin of the \((x,y)\) cartesian coordinate system, we have:

\[ x = r \cos \theta = 2a \cos^2 \theta \]

\[ y = r \sin \theta = 2a \cos \theta \sin \theta \]

so that

\[ \dot{x} = -4a \cos \theta \sin \theta \dot{\theta} = -2a \sin 2\theta \dot{\theta} \]

\[ \dot{y} = -2a (\sin^2 \theta - \cos^3 \theta) \dot{\theta} = 2a \cos 2\theta \dot{\theta} \]

and

\[ \dot{\nu}^2 = \dot{x}^2 + \dot{y}^2 = 4a^2 \dot{\theta}^2 \]

\[ \nu = 2a \dot{\theta} \]
From the angular momentum equation, we have

\[ l = m r^2 \dot{\theta} = ma^2 \cos^2 \theta \dot{\theta} \]

So,

\[ \dot{x} = -2a \sin 2\theta \left( \frac{\dot{l}}{ma^2 \cos^2 \theta} \right) \]
\[ = -\frac{2l}{ma} \frac{\sin 2\theta}{\cos^2 \theta} \]
\[ \dot{y} = \frac{2l}{ma} \frac{\cos 2\theta}{\cos^2 \theta} \]
\[ \dot{v} = \frac{2l}{ma} \frac{1}{\cos^2 \theta} \]

As \( \theta \to \pm \pi/2 \), \( \cos^2 \theta \to 0 \), so that it is obvious that both \( \dot{y} \) & \( \dot{v} \to \infty \).

For \( \dot{x} \), note that

\[ \frac{\sin 2\theta}{\cos^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta} = \frac{2 \tan \theta}{\cos \theta} \to \infty \]

as \( \theta \to \pm \pi/2 \),

\[ \Rightarrow \text{ so that all three quantities} \]
\[ x, \dot{y}, v \to \infty \text{ as } \theta \to \pm \pi/2 \text{ (as the orbit goes thru the center of force)} \]

Note that this is a unique singular case when the orbit is passing thru its center of force.

Although \( x, \dot{y}, \theta, T, V \) all blow up as the orbit approaches its center of force, the two constants of motion \( E \) & \( L \) remain finite & constant.
\[ V(r) = -\frac{k}{r} e^{-\frac{ra}{r}} \quad \text{(Yukawa potential)} \]

\( k > 0 \) and \( a > 0 \)

\[ T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \]

\[ L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} e^{-\frac{ra}{r}} \]

\[ \dot{r} - \frac{1}{r} \frac{dL}{dr} = 0 \implies m\ddot{r} - m r \dot{\theta}^2 - f(r) = 0 \quad (1) \]

where \( f(r) = -\frac{dV}{dr} = -\frac{k}{r^2} e^{-\frac{ra}{r}} - \frac{k}{ar} e^{-\frac{ra}{r}} \)

\[ \dot{\theta} - \frac{dL}{d\theta} = 0 \implies \frac{d}{dt} (m r^2 \dot{\theta}) = 0 \]

\[ \Rightarrow m r^2 \dot{\theta} = \ell \quad \text{(Angular momentum is conserved)} \]

Rewriting \( (1) \) in terms of \( \ell \), we have

\[ m \ddot{r} = \frac{\ell^2}{mr^2} + f(r) \]

If we let \( V'(r) = \frac{\ell^2}{2mr^2} + V(r) \) as an effective potential, then our equation of motion can be written as a one-dimensional problem:

\[ m \ddot{r} = -\frac{d}{dr} V'(r) = -\frac{d}{dr} \left( \frac{\ell^2}{2mr^2} + V(r) \right) \]
Now, let us use the effective potential to discuss the qualitative nature of the orbits.

\[ V'(r) = \frac{\ell^2}{2mr^2} - \frac{k}{r} e^{-r/a} \]

Consider the following cases:

1. \( \ell = 0 \) (no angular momentum):

\[ V'(r) = -\frac{k}{r} e^{-r/a} \]

For \( E \geq 0 \), the motion is unbounded!

For \( E < 0 \), the motion is bounded by \( r < r^* \).

Also, since \( \ell = 0 \) (\( \theta = 0 \)), the particle will travel along a line straight toward the center of force.

2. \( \ell \neq 0 \):

\[ V'(r) = \frac{\ell^2}{2mr^2} - \frac{k}{r} e^{-r/a} \]

Note: \( r \to 0, \)
\[ \frac{\ell^2}{2mr^2} \to +\infty \]
\[ -\frac{k}{r} e^{-r/a} \to -\infty \]

but \( \frac{\ell^2}{2mr^2} \) will dominate.

\( r \to \infty, \) \( e^{-r/a} \to 0 \) faster than \( \frac{1}{r^2} \) so that again \( \frac{\ell^2}{2mr^2} \) will dominate.
Putting these two observations together, we have

\[ \frac{e^2}{4\pi \epsilon_0 r^2} \text{ dominates} \]

At \( r \to 0 \)

\[ -\frac{k}{r} e^{-\frac{r}{a}} \text{ is important in the middle} \]

& shape of the dip will depend on the value of \( k \).

To examine how the shape of the dip is determined by \( k \),

let look at the extrema for \( V'(r) \):

Setting \( \frac{dv'}{dr} = 0 \) gives:

\[ -\frac{e^2}{mr^2} + \frac{k}{r^2} e^{-\frac{r}{a}} + \frac{k}{ra} e^{-\frac{r}{a}} = 0 \]

\[ ak^2 - mkr e^{-\frac{r}{a}} - mk^2 e^{-\frac{r}{a}} = 0 \]

\[ \frac{ak^2}{mk} = (r^2 + ar) e^{-r/a} \]

\[ \frac{e^2}{amk} = (x^2 + x)e^{-x} \quad \text{, where } x = \frac{r}{a} \]
Plotting \((x^2-x)e^{-x}\),

- we can see that

\[
\frac{e^2}{amk} < y^* = (x^* + x^*)e^{-x^*} \\
= (2 + \sqrt{2})e^{-\left(\frac{1+\sqrt{2}}{2}\right)} \\
\approx 0.84
\]

Then, there will be no solution to (2) and \(V(r)\) will not have any dips.

On the other hand, if \(\frac{e^2}{amk} < y^* \approx 0.84\), then \(V(r)\) will have two possible extrema \(\hat{r}_1 \& \hat{r}_2\) and the possibility for a dip (potential well) (see graphs above & below):

\[
\frac{e^2}{amk} < y^* \approx 0.84
\]
Note that it is possible for $V'(r)$ to be non-negative and still form a potential well.

This occurs when $V'(\hat{r}_1) \geq 0$.

For $0 < E$ (total energy) $< V'(\hat{r}_2)$, the particle will still be trapped in the potential.

Now, let consider a $l$ value such that we will have a potential well, i.e. \( \frac{L^2}{\alpha m} < \mathcal{K} \). We want to consider the possible orbits for a given $E$.

The analysis will be very similar to the ones in the book.
1. For $E > E_2$, the orbit will be unbounded.

2. For $E < E_1$, there will be no solution.

3. For $E_1 < E < E_2$, the orbit will be stably bounded by the potential moving between two radii ($r_{\text{min}}$ and $r_{\text{max}}$).

A particle from infinity will move toward the center of force and reaches its closest approach at $r_{\text{min}}$ and it will then move back out to infinity.
Note: different from the inverse-square law or hook's law potentials, the orbit in this case will not close onto itself.

4. For $E = E_1$ or $E = E_2$, there will be the possibility for circular orbits but only the $E = E_1$ case will be a stable circular orbit.