To show that the Nielsen form is equivalent to the original form of the Lagrange equations:

\[
\frac{\partial T}{\partial \dot{q}_i} - 2 \frac{\partial T}{\partial q_i} = 0_i
\]

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - 2 \frac{\partial T}{\partial q_i} = 0_i
\]

Sum over j's

\[
\frac{d}{dt} = \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial T}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial T}{\partial q_j} q_j - T(q_i, \dot{q}_i, t)
\]

Sum over j's

So, \[
\frac{\partial T}{\partial q_i} = \frac{2}{\dot{q}_i} \left( \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial T}{\partial q_j} q_j - T(q_i, \dot{q}_i, t) \right)
\]

\[
= \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{2}{\dot{q}_i} \left( \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial T}{\partial q_j} q_j - T(q_i, \dot{q}_i, t) \right)
\]

Switching order

\[
= \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{2}{\dot{q}_i} \left( \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial T}{\partial q_j} q_j - T(q_i, \dot{q}_i, t) \right)
\]

So, \[
\frac{\partial T}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{2}{\dot{q}_i} \left( \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial T}{\partial q_j} q_j - T(q_i, \dot{q}_i, t) \right)
\]

Putting this back into the Nielsen Equation, we have:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial T}{\partial q_i} - 2 \frac{\partial T}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = 0_i
\]
**Extra Problem (T&M 9.57)**

A. Show that a rocket in free space (i.e., no external force) of initial velocity $V_0$ and mass $M_0$ attains a speed of $V$ by ejecting mass (i.e., burning fuel), where $V$ is given by

$$V = V_0 + u \ln \left( \frac{M_0}{m} \right)$$

where $u$ is the exhaust velocity of the fuel and is assumed to be constant.

Since the system is in free space, within an infinitesimal time $dt$, $dP_{tot} = 0$.

$$dP_{tot} = P_{tot}(t+dt) - P_{tot}(t) \rightarrow mv$$

$$\downarrow$$

$$(m - dm)(u + dv) + dm'(u - m) \quad dm' \rightarrow \text{ejected fuel mass}$$

$$= mv + mdv - udm' + dm'udv + $2nd order

$$= mdv - udm'$$

$$dP_{tot} = 0 \Rightarrow dv = \frac{u dm'}{m}$$

by choice, mass of fuel used

$$dm = - dm' \quad \text{mass loss by rocket}$$

Integrate: $V = V_0 + u \ln \left( \frac{M_0}{m} \right)$
B. If the rocket accelerates at a constant acceleration $a$ to its final speed $v$, what is the total work done by the rocket engine?

- For constant acceleration $a$, we have

$$ dv = adt $$

- From part a, we have

$$ dv = -\mu \frac{dm}{m} \Rightarrow \quad adt = -\mu \frac{dm}{m} $$

$$ -\frac{at}{m} = \ln \left( \frac{m}{m_0} \right) $$

So,

$$ m(t) = m_0 e^{-at/m} $$

- From the definition of work, we have

$$ W = \int_{c}^{f} F \, dx = \int_{c}^{f} \frac{dp}{dt} \, dx = \int_{c}^{f} v \, dp $$
\[ W_r = \int_i^f v \, dm(v) \]
\[ W_L = \int_i^f (v-u)(-\, dm(v-u)) \]
\[ W_{tot} = W_r + W_L = \int_i^f v \, mdv + u^2 \, dm - (u-v)^2 \, dm \]
\[ = \int_i^f m \, v \, dv + u(\omega v-u) \, dm \]
\[ dv = u \, \frac{dm}{m} \quad \text{or} \quad -\frac{m}{u} \, dt = dm \]
\[ W_{tot} = \int_i^f m \, v \, dv + u(\omega v-u) \left( -\frac{m}{u} \, dv \right) \]
\[ = \int_i^f m \, v \, dv - \omega m \, v \, dv + m \, udv \]
\[ = \int_i^f (u-u) \, m_0 \, e^{-\frac{v}{u}} \, dv \quad (t_k, \omega v_0 = v) \]
\[ \text{let } x = \frac{v}{u} \quad dx = \frac{dv}{u} \]
\[ = m_0 \, u^2 \int_i^f (1-x) \, e^{-x} \, dx \]

Continues on (4)

(Integrating by parts)
Integrating by parts,

let \( e = \sqrt{1 - x} \)

\[ dt = e^{-x} \, dx \]
\[ -ds = -\, dx \]
\[ t = -e^{-x} \]

\[ W_{41f} = m_0 \mu^2 \left[ -\left(1 - x\right) e^{-x} \right]_{0}^{\frac{\nu}{\mu}} - \int_{0}^{\frac{\nu}{\mu}} e^{-x} \, dx \]

\[ = m_0 \mu^2 \left[ -\left(1 - \frac{\nu}{\mu}\right) e^{-\frac{\nu}{\mu}} + 1 + e^{-\frac{\nu}{\mu}} \right]_{0}^{\frac{\nu}{\mu}} \]

\[ = m_0 \mu^2 \left[ -e^{-\frac{\nu}{\mu}} + \frac{\nu}{\mu} e^{-\frac{\nu}{\mu}} + e^{-\frac{\nu}{\mu}} \right] \]

\[ = m_0 \nu \mu e^{-\frac{\nu}{\mu}} \]

\[ W_{41f} = m_0 \nu \mu \]