But, the advantage is that dynamics on $s$ is with one less dimension.

Minimum degrees of freedom for chaos

Flows: $N \leq 2$

Poincaré–Bendixson Theorem

types of possible motions:

1. Limit set is a point

2. Limit set is a limit cycle

3. Limiting set is the figure 8 with eq. pt. $P$ at vertex

(No Chaos possible)

*trajectories can't expand and remain bounded in 2-space! (trajectories can't cross)
Possibilities for chaos

**Flows**  \( n \geq 3 \)

**Maps**  \( n \geq 2 \) (invertible)
\( n \geq 1 \) (non-invertible)

\[ x_{n+1} = R x_n (1-x_n) \]

\[ \begin{align*}
\text{1D Logistic Map:} \\
x_{n+1} &= R x_n (1-x_n) \\
\end{align*} \]

\[ \begin{align*}
\text{2D Heron Map:} \\
x_{n+1} &= A - x_n^2 + b y_n \\
y_{n+1} &= y_n \\
\end{align*} \]
\[ X(t) \]
\[ X'(t) \]
\[ X(t) \] 
\[ X'(t) \]

**Fundamental characters of chaos**

1. **Dynamical:** Sensitivity to initial conditions

   (Definition of chaos)

   \[ |\delta X(t)\| \sim e^{ht} |\delta X(0)\| \]

   \[ \checkmark \text{If } h > 0, \text{ then error will exponentially grow} \]

   \[ \rightarrow \text{chaos} \]

   \[ \checkmark \text{If } h < 0, \text{ then error will not grow} \]

   \[ \rightarrow \text{No sensitivity to IC, not chaotic} \]

**Example:** Heron map

\[ \begin{cases} X_{n+1} = 1.4 - X_n^2 + 0.3Y_n \\ Y_{n+1} = X_n \end{cases} \]
If $|\delta x(t)| \approx 10^{-14}$ round off error after $n \geq 45$ iterations

$2^{10^{-4}} \approx 1 \quad \text{(error will grow to system size!)}$

* long term predictability is lost! *

* To put it in another way:

If we want to improve our prediction to $n = 90$ (double the time),

our initial accuracy has to be $\approx 10^{-28}$

which is 14 orders of magnitude smaller!

* This has profound implication on real system in contrast to pure mathematical models.

$(\delta x(t) \approx 0$ due to noise/QM fluctuations)
Box-Covering dimension:

\[ N(\varepsilon) = \# \text{of boxes with size } \varepsilon \text{ needed to cover a set} \]

Observation:

- Line
  \[ A = \varepsilon^1 \]
  \[ N(\varepsilon) \sim \frac{A}{\varepsilon} \]
  \[ \text{If } \varepsilon \times \frac{1}{2} \]
  \[ N(\varepsilon) \times 2^1 \]
- Area
  \[ A \]
  \[ N(\varepsilon) \sim \frac{A}{\varepsilon^2} \]
  \[ \text{If } \varepsilon \times \frac{1}{2} \]
  \[ N(\varepsilon) \times 2^2 \]

So, we can say that in general, the \# of boxes needed to cover a set should scale with its dimension!

Or, \[ N(\varepsilon) \sim \varepsilon^{-D} \]