Quantum Mechanics Qualifying Exam

Jan. 2011

You are allowed to quote results directly from the main text of Shankar, but not from its exercises or solutions.

1. A particle of mass $m$ is confined within a cubic box of side length $L$. The wave-function vanishes beyond the box.
   a). What is the parity (eigenvalue) of the ground state? Assume the origin is at the center of the box. [5 pts]
   b). What is the energy of the first excited state? What is its degeneracy? [10 pts]
   c). Assume the particle is in its ground state. Suddenly, the box expands to a cube with side $2L$. What is the probability of finding the particle in the new ground state? [10 pts]
   d). Consider 3 non-interacting identical bosons in the box, what is the energy of the first excited state? [5 pts]

2. A molecule is rotating around its center of mass. The Hamiltonian is $H = (L_x^2 + L_y^2)/2I_a + L_z^2/2I_b$, where $I_{a,b}$ are the moments of inertia, and $L_{x,y,z}$ are the orbital angular momentum operators.
   a). Find the energy eigenvalues and eigenstates. [10 pts]
   Now consider a state described by angular wave function $\psi(\theta, \phi) = \sqrt{3/4\pi} \sin \theta \cos \phi$, where $\theta$ and $\phi$ are the polar and azimuthal angles respectively. Using the Dirac ket notation $|lm\rangle$ may prove convenient for subsequent problems 2b) and 2c).
   b). Compute the expectation value of $L_z$ in state $\psi$. [10 pts]
   c). Suppose $L_z$ is measured in state $\psi$ and result $\hbar$ is obtained. Immediately afterwards, $L_x$ is measured, find the uncertainty (standard deviation) $\Delta L_x$. [10 pts]

3. In April (O’Connell et al, Nature 464, 697, 2010), a group of physicists at UCSB succeeded in quantum control of a macroscopic mechanical system. A mechanical resonator made of aluminum nitride and aluminum was cooled to 25mK. A superconducting qubit was coupled to the resonator to prepare and measure its quantum states.
   Treat the mechanical resonator as a one-dimensional harmonic oscillator of frequency $\omega$. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ be the normalized energy eigenstate of the ground state and the first excited state respectively. At time $t = 0$, the system is prepared at state $A|\psi_0\rangle + B|\psi_1\rangle$. $A$ and $B$ are in general complex numbers. Then, the dynamics of the system is monitored in the experiment to determine the relaxation and coherence time.
   a). Compute the average value of energy $E$ and position $x$ at $t = 0$. [10 pts]
   b). Find the state vector at later time $t > 0$. [10 pts]
   c). What is the oscillation frequency of $\langle x \rangle$ as a function of time? [10 pts]

4. A boy drops a marble of mass $m$ from height $H$ (above the ground) and tries to hit a marked point on the ground. Show that no matter how hard he tries, the marble is going to miss the point by a distance $\Delta x$ on the order of $(\hbar^2 H/m^2 g)^{1/4}$, where $g$ is the gravitational acceleration. [10 pts]
   Hint: the initial (horizontal) position and velocity are constrained by Heisenberg’s uncertainty relation. Treat the motion of the marble after release as classical. This problem is a mock version of $^{87}$Rb atoms released from an optical trap.