Problem 1 [26 points]

Consider a gas of identical classical particles, each of mass $m$, in thermal equilibrium at a temperature $T$. If $\mathbf{v} = (v_x, v_y, v_z)$ is the particle velocity and $v$ is its speed, calculate the following average values:

(a) [2 points] $\bar{v}_x$
(b) [4 points] $\bar{v}_x^2$
(c) [2 points] $\bar{v}_x^3$
(d) [4 points] $\bar{v}_x^2 v_x$
(e) [4 points] $\bar{v}_x^2 v_y^2$
(f) [4 points] $\bar{v}$
(g) [4 points] $\langle 1/v \rangle$

(h) [2 points] Using the results of (f) and (g), show that the general inequality $\bar{v}(1/\bar{v}) > 1$ appearing in Problem 5 is satisfied.

Problem 2 [24 points]

Consider a free electron gas at $T = 0$ K. Suppose its volume is $V$ and the number of electrons is $N$.

1. [4 points] Show that the total kinetic energy of the gas is

$$U_0 = \frac{3}{5} N \varepsilon_F,$$

where $\varepsilon_F$ is the Fermi energy.

2. [5 points] Derive the following relation between the gas pressure $p$ and total energy $U_0$:

$$pV = \frac{2}{3} U_0.$$  

(2)

3. [5 points] Show that the isothermal compressibility of the gas, $\beta_T = -(\partial \ln V/\partial p)_{T,N}$, equals

$$\beta_T = \frac{3V}{2N \varepsilon_F}.$$  

(3)

4. [5 points] The speed of sound in a gas is given by

$$v_s = [(\partial p/\partial \rho)_T]^{1/2},$$  

(4)
where $\rho$ is the gas density (mass per unit volume). Compute $v_s$ for the free electron gas at $T = 0$ K and compare it with the Fermi velocity $v_F$.

5. [5 points] If $v$ is the electron speed, calculate $\bar{v}$, $(1/v)$, and check if the general inequality $\bar{v}(1/v) > 1$ appearing in Problem 5 is satisfied.

**Problem 3 [24 points]**

Imagine a harmonic solid with an isotropic dispersion relation $\omega = Ak^b$, where $\omega$ is the angular frequency of atomic vibrations, $k$ is the wave number, and $A > 0$ and $b > 0$ are constants. Assuming that this dispersion relation holds for each of three polarizations of phonons, show that in the low-temperature limit the phonon contribution to the heat capacity of the solid is proportional to $T^{3/b}$.

**Problem 4 [26 points]**

A system has two quantum states, state 0 with energy 0 and state 1 with energy $\varepsilon$. These states can be occupied by non-interacting fermions from a particle and heat reservoir at a temperature $T$ and chemical potential $\mu$.

1. [6 points] Calculate the grand partition function $\Gamma(T, \mu)$ of the system.

2. Using the obtained $\Gamma(T, \mu)$, compute the following properties as functions of $T$ and $\mu$:

   (a) [6 points] Average occupation numbers of the two states, $\bar{n}_0$ and $\bar{n}_1$.

   (b) [6 points] Average total energy $\bar{E}$.

   (c) [8 points] The system entropy $S$.

**Extra Credit Problem**

**Problem 5 [10 points]**

Prove that for any probability distribution of classical or quantum particles

$$\bar{v}(1/v) > 1,$$

where $v$ is the particle speed.
Moments of the Gaussian function:

\[ M_n = \int_0^\infty x^n e^{-x^2} \, dx. \quad (6) \]

Selected values: \( M_0 = \sqrt{\pi}/2, \ M_1 = 1/2, \ M_2 = \sqrt{\pi}/4, \ M_3 = 1/2, \ M_4 = 3\sqrt{\pi}/8, \ M_5 = 1, \ M_6 = 15\sqrt{\pi}/16. \)