Qualifying exam - August 2013

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

**Problem 1** [20 points]
Consider a system of non-interacting identical localized oscillators. Using the classical Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2, \]

(1)

\( (m \) is the particle mass and \( x \) displacement from equilibrium) calculate

1. [10 points]

\[ \langle x^2 - \bar{x}^2 \rangle^2. \] (2)

2. [10 points]

\[ \langle p^2 - \bar{p}^2 \rangle^2. \] (3)

**Problem 2** [20 points]
Calculate the average energy per photon in black-body radiation (total energy divided by the number of photons). Show that this energy is approximately \( \varepsilon \approx 2.701kT \).

**Problem 3** [30 points]

*Two-dimensional universe!* Imagine that our universe is two-dimensional (2D). By analogy with the 3D theory of black-body radiation, develop a similar theory for a 2D universe. Specifically, consider a cavity of an area (2D “volume”) \( A \) filled with black-body radiation at a temperature \( T \). Derive the following thermodynamic properties:

1. [5 points] Helmholtz free energy \( F(T, A) \).
2. [3 points] Entropy \( S(T, A) \).
3. [3 points] Radiation pressure \( p(T) \).
4. [4 points] Energy \( E(T, A) \). Is the Stefan-Boltzmann Law still valid?
5. [4 points] Specific heat \( C_v(T, A) \).
6. [5 points] Total number of photons \( N(T, A) \).
7. [6 points] Fundamental equation of state \( S(E, A) \).

**Problem 4** [30 points]
Consider a gas in equilibrium with a solid surface containing identical adsorption sites. When a molecule adsorbs, its energy changes by \( \varepsilon < 0 \) due to chemical interaction with the surface. In addition, it acquires a magnetic moment \( m \) which can be aligned either parallel or anti-parallel to an applied magnetic field \( H \). Interaction between the adsorbed
molecules can be neglected. For given temperature $T$ and chemical potential $\mu$ in the gas, apply the grand-canonical formalism to
1. [8 points] Calculate the average fraction of surface sites occupied by molecules.
2. [7 points] Calculate the average magnetic moment $\overline{m}$ per surface site.
3. Now consider the small-field limit, i.e., $mH \ll kT$ at fixed values of $\varepsilon$ and $\mu$.
   3a. [7 points] Show that the magnetic moment of the surface is proportional to $H$.
   3b. [8 points] Find the mean-squared fluctuation $(\overline{m} - \overline{m})^2$. 
Formula Sheet

Riemann’s zeta function:
\[ \zeta(n) = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx. \]  \hfill (4)

Selected values: \( \zeta(2) = \pi^2/6, \zeta(3) \approx 1.202 \) and \( \zeta(4) = \pi^4/90. \)

Moments of the Gaussian function:
\[ M_n = \int_0^\infty x^n e^{-x^2} dx. \]  \hfill (5)

Selected values: \( M_0 = \sqrt{\pi}/2, M_1 = 1/2, M_2 = \sqrt{\pi}/4, M_3 = 1/2, M_4 = 3\sqrt{\pi}/8, M_5 = 1, M_6 = 15\sqrt{\pi}/16. \)