QM Qualifier May 2014

Problem 1 (30points)

In a two-state system, two sets of basis kets, \( |e \pm \rangle \) (denoted as e-basis) and \( |d \pm \rangle \) (d-basis) are related as \( |d \pm \rangle = \frac{1}{\sqrt{2}} (|e + \rangle \pm i|e - \rangle) \). The matrix form for a state vector \( |\alpha\rangle = \begin{pmatrix} \cos \beta \\ e^{i \varphi} \sin \beta \end{pmatrix} \) is given in d-basis and the operators of two observables are given in e-basis as, \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

(a) (10pts) Calculate the expectation value for \( A \) and \( B \).

(b) (10pts) Calculate the uncertainty for both observables.

(c) (10pts) Verify the uncertainty principle.

Problem 2 (20points)

A state for simple harmonic oscillator of frequency \( \omega \) starts (at \( t=0 \)) from an arbitrary superposition of two number states as, \( |\alpha,0\rangle = \cos \theta |n\rangle + e^{i \varphi} \sin \theta |l\rangle \), where \( \theta \) and \( \varphi \) are real and \( n > l \).

(a) (5pts) Write the state vector at time \( t \).

(b) (5pts) What is the energy expectation value at time \( t \). Is it a periodic function of time? If yes, what is the period?

(c) (10pts) Calculate the expectation value of potential energy at time \( t \). Is it a periodic function of time? If yes, what is the period?

Problem 3 (30 points)

Let \( |jm\rangle \) represent the eigenket of the angular momentum \( J^2 \) and \( J_z \) of a system. The system is in a state \( |\alpha\rangle \), which is a normalized superposition of \( |11\rangle, |10\rangle, \) and \( |1-1\rangle \). If you measure \( J_z \), the possible values are \( \hbar, 0, \) and \( -\hbar \) with equal probability.

(a) (10points) Find an explicit form of \( |\alpha\rangle \) as the superposition of the eigenket \( |jm\rangle \). Write the density matrix.

(b) (20points) Calculate expectation value \( \langle J_x \rangle, \langle J_y \rangle, \) and \( \langle J_z \rangle \).
Problem 4 (20 points)

Two spin-1/2 particles A and B are set to an entangled state, \( |\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |\rangle) \). \(+\rangle\) denotes particle A in spin-up state along z-direction while particle B in the spin-down state along z-direction.

(a) (5 points) When \( S_z \) is measured for particle A, what are the possible values and corresponding probability if B is not measured?

(b) (5 points) What results should be for the same question in (a) if particle B is measured at the same time to be in spin-up state?

(c) (10 points) Along direction \( x \), which is perpendicular to z-axis, \( S_x \) of both particles in state \( |\alpha\rangle \) are measured. If the result for particle B is \( S_x = \frac{\hbar}{2} \), what should be the result for A? Show the steps leading to your answer.