First, notice that $\langle H \rangle = -\frac{\partial}{\partial \beta}(\ln Z)$. This is evident from

$$\langle H \rangle = \int H \left(\frac{e^{-\beta H}}{Z} \right) dx\,dy\,dP_x\,dP_y$$

$$= -\frac{1}{Z}\frac{\partial}{\partial \beta}\left[\int e^{-\beta H}\, dx\,dy\,dP_x\,dP_y \right] = -\frac{1}{Z}\frac{\partial}{\partial \beta}(Z).$$

So, we evaluate Z.

$$Z = \int \exp\left(-\frac{\beta P_x^2}{2m} - \frac{\beta P_y^2}{2m} - \frac{\beta L}{2}x^2 - \frac{\beta L}{2}y^2 \right) dx\,dy\,dP_x\,dP_y$$

There should be a factor of h^(-2) in front here which carries through.

The integrals separate,

$$Z = \left(\int_{-\infty}^{\infty} e^{-\beta \frac{P_x^2}{2m}} dP_x \right) \left(\int_{-\infty}^{\infty} e^{-\frac{\beta P_y^2}{2m}} dP_y \right) \left(\int_{-\infty}^{\infty} e^{-\frac{\beta L}{2}x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{\beta L}{2}y^2} dy \right).$$

Recall that $\int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$. So,

$$Z = \left(\sqrt{\frac{2\pi m}{\beta}} \right)^2 \left(\sqrt{\frac{2\pi}{\beta L}} \right)^2 = \frac{4\pi^2 m}{\beta^2 L}.$$

Now $\ln Z = \ln\left(\frac{4\pi^2 m}{L} \right) - 2\ln\beta$

and $\langle H \rangle = -\frac{\partial}{\partial\beta}(\ln Z) = -\left(-\frac{2}{\beta} \right) = 2kT$. So $\boxed{\langle H \rangle = 2kT}$.

Now we evaluate Z in polar coordinates.

Note that

$$x = r\cos\theta \qquad p_x = m\dot{x}$$
$$y = r\sin\theta \qquad p_y = m\dot{y}$$

and so

$$p_x = -mr\sin\theta\,\dot\theta + m\dot r\cos\theta$$
$$p_y = mr\cos\theta\,\dot\theta + m\dot r\sin\theta$$

We express these in terms of $\boxed{p_r = m\dot r}$ and $\boxed{p_\theta = mr^2\dot\theta}$:

$$p_x = -(mr^2\dot\theta)\frac{\sin\theta}{r} + (m\dot r)\cos\theta \quad = -p_\theta\left(\frac{\sin\theta}{r}\right) + p_r\cos\theta$$
$$p_y = (mr^2\dot\theta)\frac{\cos\theta}{r} + (m\dot r)\sin\theta \quad = p_\theta\left(\frac{\cos\theta}{r}\right) + p_r\sin\theta$$

Summarizing, we have:

$$\left\{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ p_x &= p_r\cos\theta - p_\theta\left(\frac{\sin\theta}{r}\right) \\ p_y &= p_r\sin\theta + p_\theta\left(\frac{\cos\theta}{r}\right) \end{aligned}\right\} \quad (*)$$

So,

$$J = \frac{\partial(x, y, p_x, p_y)}{\partial(r, \theta, p_r, p_\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 & 0 \\ \sin\theta & r\cos\theta & 0 & 0 \\ \dfrac{p_\theta\sin\theta}{r^2} & -p_r\sin\theta - p_\theta\dfrac{\cos\theta}{r} & \cos\theta & -\dfrac{\sin\theta}{r} \\ -\dfrac{p_\theta\cos\theta}{r^2} & p_r\cos\theta - p_\theta\dfrac{\sin\theta}{r} & \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix}$$

Now we need $\det(J)$. Expanding along the top row,

$$\det(J) = \cos\theta \begin{vmatrix} r\cos\theta & 0 & 0 \\ -P_r\sin\theta - P_\theta\frac{\cos\theta}{r} & \cos\theta & -\frac{\sin\theta}{r} \\ P_r\cos\theta - P_\theta\frac{\sin\theta}{r} & \sin\theta & \frac{\cos\theta}{r} \end{vmatrix} - (-r\sin\theta) \begin{vmatrix} \sin\theta & 0 & 0 \\ \frac{P_\theta\sin\theta}{r^2} & \cos\theta & -\frac{\sin\theta}{r} \\ -\frac{P_\theta\cos\theta}{r^2} & \sin\theta & \frac{\cos\theta}{r} \end{vmatrix}$$

and expanding each of these 3x3 determinants along their top rows,

$$= \cos\theta \left(r\cos\theta \begin{vmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{vmatrix} \right) + r\sin\theta \left(\sin\theta \begin{vmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{vmatrix} \right)$$

$$= \left(r\cos^2\theta + r\sin^2\theta \right) \begin{vmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{vmatrix}$$

$$= r\left(\frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r} \right) = \frac{r}{r} = 1.$$

Therefore,

$$Z = \int e^{-\beta H}\, dx\, dy\, dP_x\, dP_y = \int e^{-\beta H}\, dr\, d\theta\, dP_r\, dP_\theta .$$

But first we need to express $H(x, y, P_x, P_y)$ as $H(r, \theta, P_r, P_\theta)$.

note

Because the Jacobian J has those zeros in the upper right, $\det(J)$ turns out to equal the product of the determinants of the upper left 2×2 block and the lower right 2×2 block. This means that it is OK, for this particular change of coordinates, to transform first $dx\,dy \to r\,dr\,d\theta$, and then $dp_x\,dp_y \to \frac{1}{r}\,dp_r\,dp_\theta$, as the problem suggests.

However, one should not expect this to hold in the general case.

Nevertheless, there is a theorem that states that

$$dq_1\,dq_2 \cdots dp_1\,dp_2 \cdots \longrightarrow du_1\,du_2 \cdots dv_1\,dv_2 \cdots$$

as long as the q's and p's are canonically conjugate, and the u's and v's are too.

In other words, $|J| = 1$ for such transformations.

This is easy to do with the equations marked (*) :

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{L}{2}\left(x^2+y^2\right)$$

$$H = \frac{1}{2m}\Bigg(P_r^2 \cos^2\theta - 2P_r P_\theta \frac{\cos\theta\sin\theta}{r} + \frac{P_\theta^2 \sin^2\theta}{r^2}$$

$$+ P_r^2 \sin^2\theta + 2P_r P_\theta \frac{\sin\theta\cos\theta}{r} + P_\theta^2 \frac{\cos^2\theta}{r^2}\Bigg)$$

$$+ \frac{L}{2}\left(r^2\cos^2\theta + r^2\sin^2\theta\right)$$

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2}\right) + \frac{L}{2}r^2 .$$

Now, finally,

$$Z = \int \exp\left(-\frac{\beta P_r^2}{2m} - \frac{\beta P_\theta^2}{2mr^2} - \frac{\beta L}{2}r^2\right) dr\, d\theta\, dP_r\, dP_\theta$$

$$Z = \left(\int_0^{2\pi} d\theta\right)\left(\int_{-\infty}^{\infty} e^{-\frac{\beta P_r^2}{2m}}\, dP_r\right)\left(\int_0^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{-\beta P_\theta^2}{2mr^2} - \frac{\beta L}{2}r^2\right) dr\, dP_\theta\right)$$

note that P_r ranges from $-\infty$ to $+\infty$

notice that these integrals do not separate

r ranges from 0 to ∞

$$Z = 2\pi\left(\sqrt{\frac{2\pi m}{\beta}}\right)\left[\int_0^\infty e^{-\frac{\beta L}{2}r^2}\left(\int_{-\infty}^{\infty} e^{-\frac{\beta p_\theta^2}{2mr^2}}\,dp_\theta\right)dr\right]$$

$$Z = 2\pi\sqrt{\frac{2\pi m}{\beta}}\left[\int_0^\infty e^{-\frac{\beta L}{2}r^2}\sqrt{\frac{2m\pi r^2}{\beta}}\,dr\right]$$

$$Z = 2\pi\sqrt{\frac{2\pi m}{\beta}}\sqrt{\frac{2m\pi}{\beta}}\int_0^\infty r\,e^{-\frac{\beta L}{2}r^2}\,dr$$

recall $I(n) = \int_0^\infty x^n e^{-\alpha x^2}\,dx$ and that $I(1) = \frac{1}{2\alpha}$

So

$$Z = \frac{4\pi^2 m}{\beta}\left(\frac{1}{\beta L}\right) = \frac{4\pi^2 m}{\beta^2 L}$$

in agreement with the cartesian integration.

Hence

$$\langle H \rangle = 2kT.$$

The equipartition Theorem, applied to $H(r, \theta, p_r, p_\theta)$, states That The $\frac{p_r^2}{2m}$ and The $\frac{L}{2} r^2$ term each contribute $\frac{1}{2} kT$ to The average energy. However, it has nothing to say about The $\frac{p_\theta^2}{2mr^2}$ term, because The coefficient of p_θ^2 involves $\frac{1}{r^2}$, and H contains a $\frac{L}{2} r^2$ term. So, you have to do The integral.