Homework #9 Phys 711 Fall 2008

Don't hand in

1. (McQuarrie, problem 4-6) Recall that the "magic step" used in class is

$$\sum_{N=0}^{\infty}\sum_{\{n_k\}}{}'\prod_k\left(\lambda e^{-\beta\varepsilon_k}\right)^{n_k}=\sum_{n_1=0}^{n_1^{\max}}\;\sum_{n_2=0}^{n_2^{\max}}\cdots\prod_k\left(\lambda e^{-\beta\varepsilon_k}\right)^{n_k},$$

where the summation with the prime is over all arrangements $\{n_k\}$ subject to $N=\sum n_k$.
To convince yourself of this, consider the summation

$$S=\sum_{N=0}^{\infty}\sum_{\{n_k\}}{}' x_1^{n_1}x_2^{n_2}$$

where $n_1, n_2 = 0,1,2.$ Show by directly expanding S for this simple case that the above is true, i.e., that

$$S=\prod_{K=1}^{2}\left(1+x_k+x_k^2\right).$$

Hand-in

1. Consider a two-dimensional harmonic oscillator with Hamiltonian

$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+\frac{L}{2}\left(x^2+y^2\right).$$

a. According to the principle of equipartition of energy, the average energy is $2kT.$ Show that this is indeed the case by integration in Cartesian coordinates and momenta.

b. Now transform this Hamiltonian to plane polar coordinates to get

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)+\frac{L}{2}r^2.$$

Be careful to get the correct expressions for the momenta conjugate to r and θ!

c. Show by direct integration in plane polar coordinates that $\langle E\rangle = 2kT.$ Do this by first transforming the integration over the Cartesian coordinates into integration over r and θ (you should obtain a familiar result), and then transforming the integration over p_x and p_y into integration over p_r and p_θ. Recall that

$$\int f(x,y)dxdy=\int f\left(x(u,v),y(u,v)\right)\left|J\right|dudv$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

You should observe an instance of a general theorem that states that transforming integrals from one set of canonically-conjugate generalized coordinates and momenta to another set has $|J| = 1$.

d. What does the equipartition theorem have to say about the Hamiltonian expressed in polar coordinates?