2. Establish that the probability density of finding a system in the canonical ensemble with energy E, $p(E)=\frac{1}{Z}g(E)e^{-\beta E}$ has a Gaussian shape in the vicinity of its most likely value E^*, i.e., that $p(E)=p(E^*)\exp\left\{-\frac{(E-E^*)^2}{2kT^2C_V}\right\}$, near E^*.

In class, we established that the first and 2nd moments were respectively, E^*, and kT^2C_V. For the third moment, we have $\left\langle (E-E^*)^3\right\rangle$

$=\left\langle E^3-3E^2E^*+3E(E^*)^2\right\rangle=\left\langle E^3\right\rangle-3E^*\left\langle E^2-EE^*\right\rangle-(E^*)^3$

$=\left\langle E^3\right\rangle-(E^*)^3-3E^*\sigma^2$.

In class we also found

$$\frac{\partial U}{\partial\beta}=-\frac{1}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty E^2g(E)e^{-\beta E}dE\right]+\frac{\langle E\rangle}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty Eg(E)e^{-\beta E}dE\right]=-\sigma^2.$$

Taking the derivative again with respect to β,

$$\frac{\partial^2U}{\partial\beta^2}=\frac{1}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty E^3g(E)e^{-\beta E}dE\right]-\frac{\langle E\rangle}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty E^2g(E)e^{-\beta E}dE\right]$$

$$-\frac{\langle E\rangle}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty E^2g(E)e^{-\beta E}dE\right]+\frac{\langle E\rangle^2}{Z}\left[\frac{1}{h^{3N}}\int_0^\infty Eg(E)e^{-\beta E}dE\right]$$

$=\left\langle E^3\right\rangle-2\langle E\rangle\left\langle E^2\right\rangle+\langle E\rangle^3$. Regrouping terms, we have

$=\left\langle E^3\right\rangle-\langle E\rangle^3-2\langle E\rangle\left\langle E^2\right\rangle+2\langle E\rangle\langle E\rangle^2=\left\langle E^3\right\rangle-\langle E\rangle^3-2\langle E\rangle\sigma^2$. So, the third moment can be written as $\left\langle (E-\langle E\rangle)^3\right\rangle=\frac{\partial^2U}{\partial\beta^2}-\langle E\rangle\sigma^2$.

Using $\frac{\partial}{\partial\beta}=-kT^2\frac{\partial}{\partial T}$, and the result from class,

$$\left\langle (E-\langle E\rangle)^3\right\rangle=k^2T^3\left(2C_V+T\frac{\partial C_V}{\partial T}\right)-\langle E\rangle kT^2C_V=\left(2kT+\frac{kT^2}{C_V}\frac{\partial C_V}{\partial T}-\langle E\rangle\right)kT^2C_V.$$

We also showed that $\frac{\sigma^2}{\langle E\rangle^2}\sim\frac{C_V}{N^2}$, so that $\frac{\sigma}{\langle E\rangle}\sim\frac{1}{\sqrt{N}}$, since $C_V\sim N$. For the case of the third moment, similarly, the term $\frac{\left\langle (E-\langle E\rangle)^3\right\rangle}{\langle E\rangle^3}\sim\frac{C_V}{N^3}\sim\frac{1}{N^2}$. So, for large enough ensembles, the distribution is approximately Gaussian.