2)

a) The energy levels are $0, \epsilon$, and 2ϵ.

- if all particles are in the $i=1$ state, then the average energy per particle is zero.

- if all particles are in the $i=3$ state, then the avg. energy per particle is $u = \frac{E_{TOT}}{\epsilon N} = \frac{N(2\epsilon)}{\epsilon N} = 2$.

Gives u units of ϵ

So $\boxed{u_{min} = 0, \quad u_{max} = 2}$

b) let n_i = the number of particles in the i^{th} state.

denote a configuration of the whole system by $\{n_i\}$, $i = 1, 2, \ldots$

The number of ways of realizing that configuration is

$$W = \frac{N!}{\prod_{i=1} n_i!}$$

The most probable configuration is that configuration that maximizes W.
So let's find it.

$$\ln W = \ln N! - \sum_{j=1}^{3} \ln(n_j!) \qquad \text{use stirling's approximation.}$$

$$\ln W \approx N\ln N - N - \sum_{j=1}^{3} (n_j \ln n_j - n_j) = N \ln N - \sum_{j=1}^{3} n_j \ln n_j \underbrace{- N + \sum_{j=1}^{N} n_j}_{\text{cancel}}$$

This is to be maximized subject to the constraints:

$$n_1 + n_2 + n_3 = N$$

$$0(n_1) + \epsilon n_2 + 2\epsilon n_3 = U = Nu\epsilon.$$

There are only three states, so using Lagrange multipliers is overkill.
let's do it by brute force.

$$d\ln W = -\sum_{j=1}^{N} n_j \frac{dn_j}{n_j} + \ln n_j \, dn_j$$

$$d\ln w = -\sum_{j=1}^{N} (1 + \ln n_j)\, dn_j$$

Set $d\ln w = 0$, so that

$$0 = (1 + \ln n_1)\, dn_1 + (1 + \ln n_2)\, dn_2 + (1 + \ln n_3)\, dn_3$$

Now use the constraints, which reduce to:

$$0 = dn_1 + dn_2 + dn_3 \longrightarrow dn_1 = -dn_2 - dn_3$$

$$0 = \epsilon\, dn_2 + 2\epsilon\, dn_3 \longrightarrow dn_2 = -2\, dn_3$$

Therefore

$$0 = (1 + \ln n_1)(-dn_2 - dn_3) + (1 + \ln n_2)(-2\, dn_3) + (1 + \ln n_3)\, dn_3$$

$$0 = -(1 + \ln n_1)\, dn_2 + \left[-(1 + \ln n_1) - 2(1 + \ln n_2) + (1 + \ln n_3)\right] dn_3$$

$$0 = \left[2(1 + \ln n_1) - (1 + \ln n_1) - 2(1 + \ln n_2) + (1 + \ln n_3)\right] dn_3$$

$$0 = 2\ln n_1 - \ln n_1 - 2\ln n_2 + \ln n_3$$

$$0 = \ln\left(\frac{n_1 n_3}{n_2^2}\right)$$

$$\frac{n_1 n_3}{n_2^2} = 1 \longrightarrow \boxed{\frac{n_1^*}{n_2^*} = \frac{n_2^*}{n_3^*}}$$ (inserting the * notation)

c)

$$X \equiv \frac{n_3}{n_2} = \frac{n_2}{n_1}.$$

So, $n_2 = x n_1$ and $n_3 = x^2 n_1$

combine these with the constraints.

$n_1 + n_2 + n_3 = N$	$\epsilon n_2 + 2\epsilon n_3 = N u \epsilon$
$n_1 + x n_1 + x^2 n_1 = N$	$x n_1 + 2x^2 n_1 = Nu$
$n_1 (1 + x + x^2) = N$	$n_1 (x)(1+2x) = Nu$

dividing one by the other,

$$\frac{Nu}{N} = \frac{n_1 (x)(1+2x)}{n_1 (1+x+x^2)} \Rightarrow u = \frac{(x)(1+2x)}{1+x+x^2}$$

$$u(1+x+x^2) = x + 2x^2$$

$$(u-2)x^2 + (u-1)x + u = 0$$

so

$$\boxed{x = \frac{-(u-1) \pm \sqrt{(u-1)^2 - 4(u-2)u}}{2(u-2)}}$$

Since x is a ratio of positive numbers (or zeros), choose the $-$ sign.

Then, as u goes from $0 \to 1 \to 2$, x goes from $0 \to 1 \to +\infty$.

d) entropy $S = k \ln W$ where W is evaluated at n_1^*, n_2^*, n_3^*.

so $$\frac{S}{k} = \ln W \doteq N \ln N - N - n_1 \ln n_1 + n_1 - n_2 \ln n_2 + n_2 - n_3 \ln n_3 + n_3$$

$$\frac{S}{Nk} = \ln N - \left(\frac{n_1}{N}\right)\ln n_1 - \left(\frac{n_2}{N}\right)\ln n_2 - \left(\frac{n_3}{N}\right)\ln n_3 .$$

recall: $n_2 = x n_1$, $n_3 = x^2 n_1$. so,

$$\frac{S}{Nk} = \ln N - \left(\frac{n_1}{N}\right)\ln n_1 - x\left(\frac{n_1}{N}\right)\ln(x n_1) - x^2\left(\frac{n_1}{N}\right)\ln(x^2 n_1)$$

$$\frac{S}{Nk} = \ln N - \ln n_1\left[\frac{n_1}{N} + \frac{x n_1}{N} + \frac{x^2 n_1}{N}\right] - x\left(\frac{n_1}{N}\right)\ln x - x^2\left(\frac{n_1}{N}\right)\ln(x^2)$$

$$\frac{S}{Nk} = \ln N - \ln n_1\left(\frac{N}{N}\right) - \ln x\left[\frac{x n_1}{N} + \frac{2x^2 n_1}{N}\right]$$

$$\frac{S}{Nk} = \ln N - \ln n_1 - \ln x\left[\frac{(n_1)(x)(1+2x)}{N}\right]$$

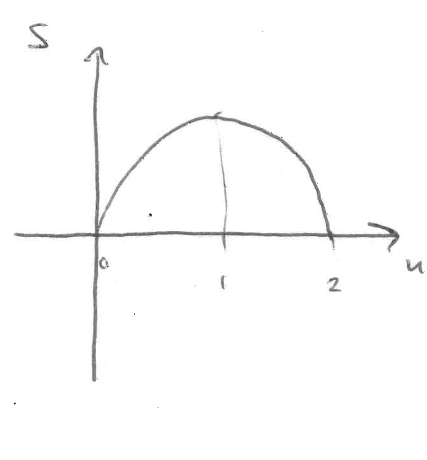
recall from constraint #2 that $(n_1)(x)(1+2x) = Nu$.

so $$\frac{S}{N} = k\left[\ln\left(\frac{N}{n_1}\right) - u \ln x\right]$$

recall from constraint #1 that $(1+x+x^2) = \frac{N}{n_1}$.

$$\boxed{\frac{S}{N} = k\left[\ln(1+x+x^2) - u \ln x\right]}$$

u	x	S
0	0	0
1	1	$k \ln 3$
2	$+\infty$	$\approx \ln\left(\frac{x^2}{x^2}\right) \to 0$



e) use $\dfrac{1}{T} = \left.\dfrac{\partial S}{\partial U}\right|_V$ where $U = Nu\epsilon$.

so $\dfrac{1}{T} = \dfrac{\partial S}{\partial u}\dfrac{\partial u}{\partial U} = \dfrac{\partial S}{\partial u}\left(\dfrac{1}{N\epsilon}\right)$.

$$\frac{N\epsilon}{T} = \frac{\partial S}{\partial u} = \frac{\partial}{\partial u}\left[Nk\left(\ln(1+x+x^2) - u\ln x\right)\right]$$

$$\frac{N\epsilon}{T} = Nk\left[\left(\frac{1+2x}{1+x+x^2}\right)\frac{\partial x}{\partial u} - \frac{u}{x}\frac{dx}{du} - \ln x\right]$$

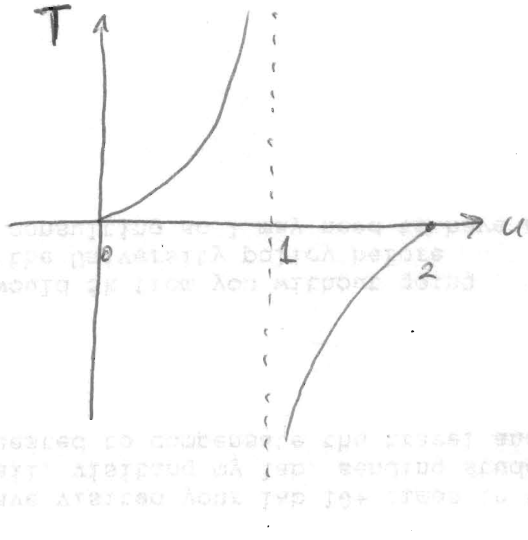
again, use $1+x+x^2 = \dfrac{N}{n_1}$ & $(n_1)(x)(1+2x) = N$

$$\frac{\epsilon}{kT} = \left(\frac{Nu}{n_1 x}\right)\left(\frac{1}{N/n_1}\right) - \frac{u}{x}\frac{dx}{du} - \ln x$$

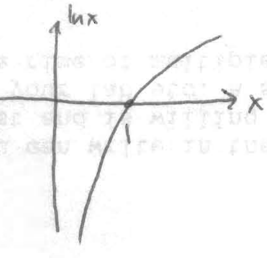
$$\frac{\epsilon}{kT} = \left(\frac{u}{x} - \frac{u}{x}\right)\frac{dx}{du} - \ln x \rightarrow \boxed{T = -\frac{\epsilon}{k\ln x}}$$

f)

u	x	T
0	0	0
1	1	$\pm\infty$
2	$+\infty$	0



recall:



9) From $T = -\dfrac{\epsilon}{k \ln x}$, get $x = e^{-\epsilon/kT}$.

recall constraints; and recall $\epsilon_i = (i-1)\epsilon$.

$$\frac{n_1}{N} = \frac{1}{1+x+x^2} = \frac{1}{1+e^{-\epsilon/kT}+e^{-2\epsilon/kT}} = \frac{1}{\sum_{j=1}^{3} e^{-\epsilon_i/kT}}$$

$n_2 = x n_1$, so

$$\frac{n_2}{N} = \frac{e^{-\epsilon/kT}}{\sum_{j=1}^{3} e^{-\epsilon_i/kT}} = \frac{e^{-\epsilon_2/kT}}{\sum_{j=1}^{3} e^{-\epsilon_i/kT}}$$

$n_3 = x^2 n_1$, so

$$\frac{n_3}{N} = \frac{e^{-2\epsilon/kT}}{N} = \frac{e^{-\epsilon_3/kT}}{\sum_{j=1}^{3} e^{-\epsilon_i/kT}}$$

Comments:

notice that as U increases from 0 to 1, entropy increases and Temperature increases. But as U increases from 1 to 2, the entropy decreases, since there are no higher energy levels. So the material gets more ordered. As a result, the Temperature is negative.