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a) 1^{st} law: $du = \delta Q + \delta W$

for the rubber band, $\delta W = \tau\, dL$

Tension — change in length.

So

$$\delta Q = du - \delta W$$

$\delta Q = du - \tau\, dL$. Given $u = C L_0 T$, we have $du = C L_0\, dT$.

So $\delta Q = C L_0\, dT - \tau\, dL$, and with $\delta Q = T\,dS$,

$$dS = \frac{\delta Q}{T} = C L_0 \frac{dT}{T} - \frac{\tau}{T}\, dL .$$

But, S is a state variable, so dS is exact. This requires:

$$\frac{\partial}{\partial L}\left(\frac{C L_0}{T}\right)_T = \frac{\partial}{\partial T}\left(-\frac{\tau}{T}\right)_L .$$

So, using $\tau = b T^n \left(\frac{L - L_0}{L_1 - L_0}\right)$,

$$0 = \frac{\partial}{\partial T}\left[-b T^{n-1}\left(\frac{L - L_0}{L_1 - L_0}\right)\right]$$

$$0 = b(n-1) T^{n-2}\left(\frac{L - L_0}{L_1 - L_0}\right)$$

Thus, $\underline{\underline{n = 1}}$.

1.b. use the 1^{st} law: $du = \delta w + \delta Q$.

The stretch is done at constant T, and we're given $u = cL_0 T$.

So $du = 0$,

and so $\delta Q = -\delta w = -\tau dL$

$$\boxed{\delta Q = -bT\left(\frac{L-L_0}{L_1-L_0}\right)dL.}$$

1.c.

Now, from before:

$$dS = \frac{cL_0}{T}dT - \frac{\tau}{T}dL$$

$$dS = cL_0\frac{dT}{T} - b\left(\frac{T}{T}\right)\left(\frac{L-L_0}{L_1-L_0}\right)dL.$$

↑
T cancels out.

integrating,

$$S - S_0 = cL_0 \ln\left(\frac{T}{T_0}\right) - \left(\frac{b}{L_1-L_0}\right)\int_{L_0}^{L}(L-L_0)\,dL$$

and

$$S = S_0 + cL_0 \ln\left(\frac{T}{T_0}\right) - \frac{b}{2(L-L_0)}(L-L_0)^2$$

So $S \propto (L-L_0)^2$.

2. a) For $u = u(T, v)$,

$$du = \left.\frac{\partial u}{\partial T}\right|_v dT + \left.\frac{\partial u}{\partial v}\right|_T dv .$$

By the 1st law, $du = \delta Q - p\,dv$.

So, $$\delta Q = \left.\frac{\partial u}{\partial T}\right|_v dT + \left(p + \left.\frac{\partial u}{\partial v}\right|_T\right) dv.$$

note that $C_v\,dT \equiv \delta Q|_v$. So, setting $dv = 0$ above,

$$C_v\,dT = \left.\frac{\partial u}{\partial T}\right|_v dT, \text{ or } \quad C_v = \left.\frac{\partial u}{\partial T}\right|_v .$$

Now, it is given that $C_v = \frac{b}{T}$.

So, $$\frac{b}{T} = \left.\frac{\partial u}{\partial T}\right|_v .$$

$$u = \int b \frac{dT}{T} = b \ln T + c, \text{ at constant volume.}$$

So $$\boxed{u = b \ln T + c}$$

c = some constant.
This is valid if volume is constant.

b) Assume the volumes remain fixed.

Then, balancing energy, $\Delta u_1 = -\Delta u_2$.

$$(b \ln T_F + c) - (b \ln T_1 + c) = -\left[(b \ln T_F + c) - (b \ln T_2 + c)\right]$$

$$2b \ln T_F = b \ln T_2 + b \ln T_1$$

$$\Rightarrow \boxed{T_F = \sqrt{T_1 T_2}}$$

3. a) $$ds = \frac{\partial s}{\partial u}\Big|_v du + \frac{\partial s}{\partial v}\Big|_u dv$$

but from the 1st law, $du = T\,ds - p\,dv$ for fixed N, so

$$ds = \frac{1}{T} du + \frac{p}{T} dv$$

hence

$$\frac{1}{T} = \frac{\partial s}{\partial u}\Big|_v \quad \text{and} \quad \frac{p}{T} = \frac{\partial s}{\partial v}\Big|_u$$

b) By analogy with the standard Legendre transformation:

$$f = f(x) \; ; \quad p = \frac{df}{dx} \; , \quad g(p) = f - px$$

we have

$$s = s(u,v) \; , \quad m = \frac{\partial s}{\partial u}\Big|_v \; \text{and} \; q = \frac{\partial s}{\partial v}\Big|_u$$

$$dB = ds - m\,du - u\,dm - q\,dv - v\,dq$$

$$dB = (m\,du + q\,dv) - m\,du - u\,dm - q\,dv - v\,dq$$

$$\boxed{dB = -u\,dm - v\,dq}$$

So

$$\boxed{B(m,q) = s - mu - qv}$$

c) Plug the given s into the above formula.

$$B = s - mu - qv = Nk s_0 + Nk \ln\left(\frac{u}{u_0}\right)^{3/2} + Nk \ln\left(\frac{v}{v_0}\right) - mu - qv$$

first, using $pv = NkT$, express

$$Nk \ln\left(\frac{v}{v_0}\right) - qv \quad \text{as} \quad Nk \ln\left(\frac{NkT}{p}\frac{p_0}{NkT_0}\right) - q\frac{NkT}{p}$$

$$= Nk \ln\left(\frac{T}{T_0}\frac{p_0}{p}\right) - q\frac{Nk}{q}$$

$$= Nk \ln\left(\frac{q_0}{q}\right) - Nk$$

Second, using $U = \frac{3}{2} NKT$, express

$$NK \ln \left(\frac{U}{U_0}\right)^{3/2} - mU \quad \text{as}$$

$$NK \ln \left(\frac{\frac{3}{2} NKT}{\frac{3}{2} NKT_0}\right)^{3/2} - m \frac{3}{2} NKT$$

$$NK \ln \left(\frac{m_0}{m}\right)^{3/2} - \frac{3}{2} NK$$

hence

$$B = B(m,q) = NKS_0 + NK \ln\left(\frac{m_0}{m}\right) - \frac{3}{2} NK + NK \ln\left(\frac{q_0}{q}\right) - NK$$

$$\boxed{B = NK\left[S_0 - \frac{5}{2} + \ln\left\{\left(\frac{m_0}{m}\right)^{3/2}\left(\frac{q_0}{q}\right)\right\}\right]}$$

d) $dB = -U\,dm - V\,dq$

$$dB = \left.\frac{\partial B}{\partial m}\right|_q dm + \left.\frac{\partial B}{\partial q}\right|_m dq$$

So

$$\left.\frac{\partial B}{\partial m}\right|_q = -U \quad \text{and} \quad \left.\frac{\partial B}{\partial q}\right|_m = -V$$

So,

$$-u = \frac{\partial B}{\partial m}\Big|_q = Nk \frac{\partial}{\partial m}\left(\ln\left(\frac{m_0}{m}\right)^{3/2}\right) = Nk\,\frac{3}{2}\,\frac{\partial}{\partial m}\left(\ln m_0 - \ln m\right)$$

$$-u = -\frac{3}{2} Nk \frac{1}{m}$$

which is $\boxed{u = +\frac{3}{2} NkT}$

also,

$$-V = \frac{\partial B}{\partial q}\Big|_m = Nk \frac{\partial}{\partial q}\left(\ln \frac{q_0}{q}\right) = -Nk\frac{1}{q}$$

$$-V = -Nk\left(\frac{T}{p}\right)$$

$$\boxed{pV = NkT}$$