5.

$$F(x,y,z)=0.$$

So

$$dF = \left.\frac{\partial F}{\partial x}\right|_{y,z} dx + \left.\frac{\partial F}{\partial y}\right|_{x,z} dy + \left.\frac{\partial F}{\partial z}\right|_{x,y} dz = 0.$$

- First, solve for $\left.\frac{\partial x}{\partial y}\right|_z$: if $z = \text{const}$, set $dz = 0$ in the above equation.

get

$$\left.\frac{\partial f}{\partial x}\right|_{y,z} dx = -\left.\frac{\partial f}{\partial y}\right|_{x,z} dy$$

so that

$$\left.\frac{\partial x}{\partial y}\right|_z = \frac{-\left.\frac{\partial f}{\partial y}\right|_{x,z}}{\left.\frac{\partial f}{\partial x}\right|_{y,z}}$$

where $\frac{dx}{dy} \rightarrow \frac{\partial x}{\partial y}$ since we set $dz = 0$.

- Similarly, one obtains

$$\left.\frac{\partial y}{\partial z}\right|_x = \frac{-\left.\partial f/\partial z\right|_{x,y}}{\left.\frac{\partial f}{\partial y}\right|_{x,z}} \quad \text{and} \quad \left.\frac{\partial z}{\partial x}\right|_y = \frac{-\left.\frac{\partial f}{\partial x}\right|_{z,y}}{\left.\frac{\partial f}{\partial z}\right|_{x,y}}.$$

- Finally, multiply these together!

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = \left(\frac{-\left.\frac{\partial f}{\partial y}\right|_{x,z}}{\left.\frac{\partial f}{\partial x}\right|_{y,z}}\right)\left(\frac{-\left.\frac{\partial f}{\partial z}\right|_{x,y}}{\left.\frac{\partial f}{\partial y}\right|_{x,z}}\right)\left(\frac{-\left.\frac{\partial f}{\partial x}\right|_{z,y}}{\left.\frac{\partial f}{\partial z}\right|_{x,y}}\right) = -1.$$

QED.

6.

- We want to write $T\,ds = c_v\,dT + T\left(\frac{\partial p}{\partial T}\right)_v dv$ in terms of α, κ_T & κ_s.

- apply the lemma to $\left(\frac{\partial p}{\partial T}\right)_v$:

$$\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_p \left(\frac{\partial v}{\partial p}\right)_T = -1.$$

So

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{-1}{\left(\frac{\partial T}{\partial v}\right)_p \left(\frac{\partial v}{\partial p}\right)_T} = -\frac{\left(\frac{\partial v}{\partial T}\right)_p}{\left(\frac{\partial v}{\partial p}\right)_T} = \frac{V\alpha}{V\kappa_T} = \frac{\alpha}{\kappa_T}$$

Thus,

$$\boxed{T\,ds = c_v\,dT + \frac{T\alpha}{\kappa_T}\,dv}$$

now, from the previous line,

$$-\left(\frac{\partial v}{\partial T}\right)_p = \frac{\alpha}{\kappa_T}\left(\frac{\partial v}{\partial p}\right)_T$$

$$\left(\frac{\partial v}{\partial T}\right)_p = -\frac{\alpha}{\kappa_T}\left(-V\kappa_T\right) = V\alpha$$

So

$$T\,ds = c_p\,dT - T\left(\frac{\partial v}{\partial T}\right)_p dp$$

becomes

$$\boxed{T\,ds = c_p\,dT - TV\alpha\,dp}$$

c) from the TdS equations

$$TdS = c_v\,dT + \frac{\alpha T}{k_T}\,dv = c_p\,dT - \alpha T v\,dp$$

$$(c_p - c_v)\,dT = \left(\frac{\alpha T}{k_T}\right)dv + (\alpha T v)\,dp$$

let's express dT in terms of dv & dp, i.e. let $T = T(v,p)$

so

$$dT = \left(\frac{\partial T}{\partial v}\right)_p dv + \left(\frac{\partial T}{\partial p}\right)_v dp$$

Then

$$(c_p - c_v)\left[\left(\frac{\partial T}{\partial v}\right)_p dv + \left(\frac{\partial T}{\partial p}\right)_v dp\right] = \left(\frac{\alpha T}{k_T}\right)dv + (\alpha T v)\,dp$$

$$\left[(c_p - c_v)\left(\frac{\partial T}{\partial v}\right)_p - \frac{\alpha T}{k_T}\right]dv = \left[\alpha T v - (c_p - c_v)\left(\frac{\partial T}{\partial p}\right)_v\right]dp$$

Since v & p are our independent variables, these coefficients must be zero if this equality is always true.

from the first,

$$c_p - c_v = \frac{\alpha T}{k_T}\,\frac{1}{\left(\frac{\partial T}{\partial v}\right)_p}$$

and from the Lemma,

$$\left(\frac{\partial T}{\partial v}\right)_p = \frac{-1}{\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial v}{\partial p}\right)_T} = \frac{-1}{\left(\frac{\alpha}{k_T}\right)(-v k_T)} = \frac{1}{\alpha v}$$

So $$c_p - c_v = \frac{\alpha T}{k_T}\alpha V = \boxed{\frac{TV\alpha^2}{k_T} = c_p - c_v}$$

d) considering an adiabatic process, so that $dS=0$,

The TdS equations are

$$0 = c_v\, dT + \frac{\alpha T}{k_T}\, dv$$

and

$$0 = c_p\, dT - \alpha T V\, dp$$

so that

$$\frac{c_p}{c_v} = \left.\frac{\alpha T V\, dp}{-\frac{\alpha T}{k_T}\, dV}\right|_s = \frac{-k_T V}{\left(\frac{\partial V}{\partial p}\right)_s} = \frac{-k_T V}{-k_s V} = \frac{k_T}{k_s}$$

so

$$\boxed{\frac{c_p}{c_v} = \frac{k_T}{k_s}}$$