1. From the first law, $du = \delta w + \delta Q$.

Consider an infinitesimal reversible process, so that $\delta Q = T ds$ and such that $\delta w = -p dv$.

Then,
$$T dS = du + p dV.$$

Now, let $u = u(T, p)$. Then
$$du = \left.\frac{\partial u}{\partial T}\right|_p dT + \left.\frac{\partial u}{\partial p}\right|_T dp.$$

Hence
$$T ds = \left.\frac{\partial u}{\partial T}\right|_p dT + \left.\frac{\partial u}{\partial p}\right|_T dp + p dV.$$

But we should think of $V = V(T, p)$ also. Hence
$$dV = \left.\frac{\partial V}{\partial T}\right|_p dT + \left.\frac{\partial V}{\partial p}\right|_T dp.$$

and so
$$T ds = \left.\frac{\partial u}{\partial T}\right|_p dT + \left.\frac{\partial u}{\partial p}\right|_T dp + p\left[\left.\frac{\partial V}{\partial T}\right|_p dT + \left.\frac{\partial V}{\partial p}\right|_T dp\right]$$

$$\boxed{T ds = \left[\left.\frac{\partial u}{\partial T}\right|_p + p\left.\frac{\partial V}{\partial T}\right|_p\right] dT + \left[\left.\frac{\partial u}{\partial p}\right|_T + p\left.\frac{\partial V}{\partial p}\right|_T\right] dp} \quad (*)$$

Notice that $\delta Q = T ds$

and that $\left.\delta Q\right|_p \equiv C_p dT$. So, setting $dP = 0$ in the above expression,

$$C_p = \left.\frac{\partial u}{\partial T}\right|_p + p\left.\frac{\partial V}{\partial T}\right|_p \qquad (1)$$

Also notice that ⊛ is

$$ds = \frac{1}{T}\left[\frac{\partial u}{\partial T}\Big|_p + p\,\frac{\partial v}{\partial T}\Big|_p \right] dT + \frac{1}{T}\left[\frac{\partial u}{\partial p}\Big|_T + p\,\frac{dV}{\partial p}\Big|_T \right] dp.$$

Since ds is an exact differential,

$$\frac{\partial}{\partial p}\left\{ \frac{1}{T}\left[\frac{\partial u}{\partial T}\Big|_p + p\,\frac{\partial v}{\partial T}\Big|_p \right] \right\}_T = \frac{\partial}{\partial T}\left\{ \frac{1}{T}\underbrace{\left[\frac{\partial u}{\partial p}\Big|_T + p\,\frac{\partial v}{\partial p}\Big|_T \right]}_{[*]} \right\}_p$$

use this to solve for a nice expression for [*].

After a fair amout of manipulation, one obtains

$$[*] = \frac{\partial u}{\partial p}\Big|_T + p\,\frac{\partial v}{\partial p}\Big|_T = -T\,\frac{\partial v}{\partial T}\Big|_p . \qquad (2)$$

So, finally, combine (1) and (2) with ⊛ :

$$\boxed{ds = c_p\, dT - T\,\frac{\partial v}{\partial T}\Big|_p\, dp.}$$

② An easier way:

First, $\delta Q \big|_p \equiv C_p \, dT$

$$T\,dS \big|_p = C_p \, dT \quad \longrightarrow \quad C_p = T \frac{\partial S}{\partial T}\bigg|_p .$$

So, with $S = S(T, P)$,

$$dS = \frac{\partial S}{\partial T}\bigg|_P dT + \frac{\partial S}{\partial P}\bigg|_T dP$$

$$dS = \frac{C_P}{T} dT + \left(\frac{\partial S}{\partial P}\bigg|_T \right) dP.$$

From the maxwell chart, you can read off:

$$\begin{array}{ccc} V & F & T \\ U & & G \\ S & H & P \end{array} \qquad \frac{\partial S}{\partial P}\bigg|_T = - \frac{\partial V}{\partial T}\bigg|_P$$

Hence

$$\boxed{T\,dS = C_p \, dT - T \frac{\partial V}{\partial T}\bigg|_P dP.}$$