a)

1. we need to add up the probability that exists in a spherical shell of radius $v = |\vec{v}|$ in order to obtain the distribution for the speed.

That's most conveniently done by changing coordinates to spherical coordinates, and integrating over the two angles.

let the spherical coordinates be v, ϕ, and θ.

Then, the answer is

$$F(v) = \iiint f(\vec{v})\, v^2 \sin\phi \, dv\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}\, v^2 \sin\phi \, dv\, d\theta\, d\phi$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}\, v^2 (2\pi) \underbrace{\left(\int_0^{\pi} \sin\phi \, d\phi\right)}_{=2}$$

$$\boxed{F(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}}$$

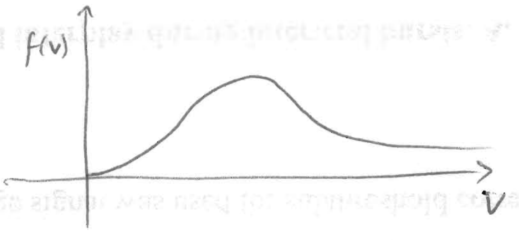
1 b) The most probable speed V_p is the speed at which $f(v)$ has a maximum. So set

$$\left.\frac{df(v)}{dv}\right|_{V_p} = 0 = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left[v^2 \left(\frac{-m}{2kT}\right) 2v\, e^{-\frac{mv^2}{kT}} + 2v\, e^{-\frac{mv^2}{kT}} \right]_{V_p}$$

$$0 = \frac{-m}{kT} V_p^3 + 2V_p$$

$$\boxed{V_p = \sqrt{\frac{2kT}{m}}}$$

One can check that this is indeed a maximum, and also that $V=0$ is a minimum. So the shape of the distribution is like this:



The average speed $\bar{V}$ is given by

$$\bar{V} = \int_0^\infty v f(v)\, dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 e^{-\frac{mv^2}{2kT}}\, dv$$

Recall the definition $I(n) = \int_0^\infty x^n e^{-ax^2} dx$

we had: $I(0) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, $I(1) = \frac{1}{2a}$, and $I(n) = -\frac{\partial}{\partial a}\left[I(n-2)\right]$.

The integral of interest is $I(3)$ with $a = \frac{m}{2kT}$.

This equals $-\frac{\partial}{\partial a}\left[I(1)\right] = -\frac{\partial}{\partial a}\left[\frac{1}{2a}\right] = \frac{1}{2a^2} = \frac{4k^2T^2}{2m^2} = 2\left(\frac{kT}{m}\right)^2$.

Hence

$$\bar{v} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[2\left(\frac{kT}{m}\right)^2\right]$$

After manipulation, $\boxed{\bar{v} = \sqrt{\frac{8kT}{\pi m}}}$.

Finally, $v_{rms} = \sqrt{\overline{v^2}}$. But this is easy, since we already know

$$\langle E_{KIN}\rangle = \frac{3}{2}kT$$

$$\langle \tfrac{1}{2}mv^2\rangle = \frac{m}{2}\langle v^2\rangle = \frac{3}{2}kT$$

So $\overline{v^2} = \frac{3kT}{m}$

and $\boxed{v_{rms} = \sqrt{\frac{3kT}{m}}}$

Just for kicks, we can obtain $\langle v^2 \rangle = \frac{3kT}{m}$ as follows:

$$\langle v^2 \rangle = \int_0^{\infty} v^2 f(v)\,dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^{\infty} v^4 e^{-\frac{mv^2}{2kT}}\,dv$$

with $a = \frac{m}{2kT}$, that integral is

$$I(4) = -\frac{\partial}{\partial a}\left(I(2)\right) = -\frac{\partial}{\partial a}\left[-\frac{\partial}{\partial a} I(0)\right] = -\frac{\partial}{\partial a}\left[-\frac{\partial}{\partial a}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)\right]$$

$$= -\frac{\partial}{\partial a}\left(\frac{\sqrt{\pi}}{4} a^{-3/2}\right) = \frac{3\sqrt{\pi}}{8}\left(\frac{m}{2kT}\right)^{-5/2}.$$

So,

$$\langle v^2 \rangle = \frac{12}{8} \frac{\pi^{3/2}}{\pi^{3/2}} \left(\frac{m}{2kT}\right)^{3/2 - 5/2} = \frac{3}{2}\left(\frac{2kT}{m}\right) = \frac{3kT}{m}$$

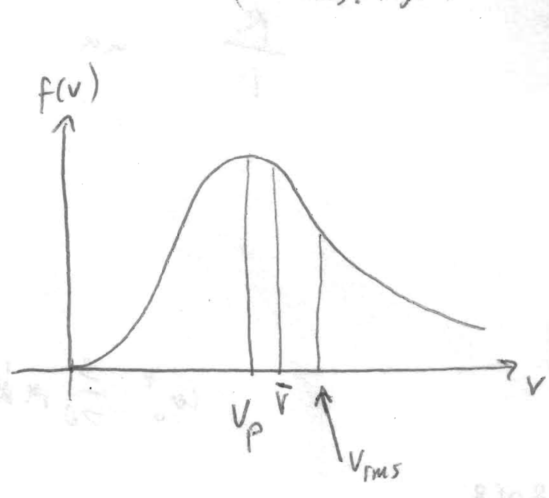
$$\sqrt{\frac{kT}{m}} = \sqrt{\frac{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{(6.65 \times 10^{-27}\ \text{kg})}} = 789.0224\ldots\ \text{m/s}$$

and

$$v_p = \sqrt{2}\left(\sqrt{\frac{kT}{m}}\right) = 1115.8\ \text{m/s} \simeq 1.12 \times 10^3\ \text{m/s}$$ (to appropriate sig. figs)

$$\bar{v} = \sqrt{\frac{8}{\pi}}\left(\sqrt{\frac{kT}{m}}\right) = 1.26 \times 10^3\ \text{m/s}$$

$$v_{rms} = \sqrt{3}\left(\sqrt{\frac{kT}{m}}\right) = 1.37 \times 10^3\ \text{m/s}$$



by the way, the units work out like this:

$$\sqrt{\frac{kT}{m}} \Rightarrow \sqrt{\frac{\left(\frac{J}{K}\right)(K)}{kg}} \Rightarrow \sqrt{\frac{J}{kg}} \Rightarrow \sqrt{\frac{Nm}{kg}} \Rightarrow \sqrt{\frac{kg\,m}{s^2}\,\frac{m}{kg}} \Rightarrow m/s.$$

c) The window width of 40 m/s is small, so

estimate $f(v)\,dv \approx f(v)\,\Delta v$

So, # of atoms in $[v_p,\ v_p + 40\ m/s]$ is approximately N_1, where

$$N_1 = N f(v_p)\,(40\ m/s)$$

$$N_1 = N 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v_p^2\, e^{-\frac{m v_p^2}{2kT}}\,(40\ m/s)$$

$$N_1 = N 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{2kT}{m}\, \exp\left(-\frac{m}{2kT}\,\frac{2kT}{m}\right)(40\ m/s)$$

$$N_1 = N 4 \left[\sqrt{\frac{m}{2\pi kT}}\, e^{-1}\,(40\ m/s)\right]$$

$$N_1 = N 4 \sqrt{\frac{6.65\times10^{-27}\ kg}{2\pi\,(1.38\times10^{-23}\ J/K)(300\ K)}}\; e^{-1}\,(40\ m/s) \qquad \text{with } N = 10^6$$

$N_1 \approx 30{,}000$ atoms

and The # of particles in $[10v_p, 10v_p + 40\,m/s]$ is approximately N_2, where

$$N_2 = N f(10v_p)(40\,m/s)$$

$$N_2 = N 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left[10\sqrt{\frac{2kT}{m}}\right]^2 \exp\left(-\frac{m}{2kT}\left[10\sqrt{\frac{2kT}{m}}\right]^2\right)(40\,m/s)$$

$$N_2 = N(100)\sqrt{\frac{m}{2\pi kT}}\, e^{-100}\, (40\,m/s)$$

$$N_2 = \frac{100}{4} e^{-99}\left[4N\sqrt{\frac{m}{2\pi kT}}\, e^{-1}(40\,m/s)\right] = 25\, e^{-99} N_1$$

which indicates that N_2 is about 10^{-42} times the answer above!

Actually, $N_2 = 7.6 \times 10^{-38}$.

Which is to say that there are essentially __NO__ particles that move at speeds of $10v_p$.

2. $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-1/2} dx$. let $n^2 = x$, so $2n\,dn = dx$ and $\frac{dx}{n} = \frac{dx}{\sqrt{x}} = 2dn$

$$= 2\int_0^\infty e^{-n^2} dn$$

$$= 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}. \quad \text{easy!}$$

Recall that

$$I = \int_{-\infty}^\infty e^{-ax^2} dx \quad \text{can be integrated like this:}$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-ax^2} dx\right)\left(\int_{-\infty}^\infty e^{-ay^2} dy\right) = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-a(x^2+y^2)} dx\,dy.$$

now let $r^2 = x^2 + y^2$ and change to polar coordinates!

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-ar^2} r\,dr\,d\theta \quad \text{integrating over the whole plane.}$$

let $u = e^{-ar^2}$. $du = -2are^{-ar^2} dr$.

so $I^2 = 2\pi \int_0^\infty e^{-ar^2} r\,dr = \frac{2\pi}{(-2a)} \int_0^\infty d\left(e^{-ar^2}\right) = -\frac{\pi}{a}\left(e^{-ar^2}\right)_0^\infty$

$$I^2 = -\frac{\pi}{a}(0-1) = \pi/a.$$

So $I = \sqrt{\pi/a}$.

Note: $\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ obviously. But if you work it out as above, be careful to integrate over just the 1^{st} quadrant of the plane!

③ $I(n) = \int_0^\infty e^{-ax^2} x^n \, dx$.

$$I(0) = \int_0^\infty e^{-ax^2} dx = \boxed{\frac{1}{2}\sqrt{\frac{\pi}{a}}}$$ (see solution to ② if needed.)

$$I(1) = \int_0^\infty e^{-ax^2} x \, dx .$$ let $y = ax^2$, so $dy = ax \, dx$.

$$I(1) = \frac{1}{2a}\int_0^\infty e^{-y} dy = -\frac{1}{2a}\left(e^{-y}\right)_0^\infty = -\frac{1}{2a}(0-1) = \boxed{\frac{1}{2a}} .$$

Now,
$$\frac{\partial}{\partial \alpha}\left[\int_0^\infty e^{-ax^2} x^{n-2} dx\right] = \int_0^\infty e^{-ax^2} (-x^2) x^{n-2} dx$$
$$= -\int_0^\infty e^{-ax^2} x^n dx.$$

So
$$\frac{\partial}{\partial a}\left(I(n-2)\right) = -I(n)$$
or
$$\boxed{I(n) = -\frac{\partial}{\partial a}\left[I(n-2)\right]} .$$

So, now we can get $I(n)$ for any positive integer.