Bean Counting

I think that the second and third sentences on the top of page 148 of Greiner are misleading. Please ignore these sentences. Read the following instead, but note that I will talk about beans instead of ensemble members.

Suppose that there is a bag containing N beans. You pick out one bean and then throw it into a set of k bins that are labeled by the index $i = 1, 2, 3, \ldots, k$. Let ω_i be the probability that the bean lands in bin i. In general, all the ω_i's are different, but it must be true that $\sum_{i=1}^{k} \omega_i = 1$. This just says that the bean must land somewhere among the bins.

Let us assume that successive bean throws are statistically independent events.

Here are several different questions that one may ask:

1. Suppose you throw q beans, one by one, into the bins, where $q \le N$. What is the probability of finding all q beans in bin i? Answer: $(\omega_i)^q$.

2. Suppose you throw all N beans, one by one, into the bins. How many beans should you expect to find in bin i? Answer: $N\omega_i$.

3. Suppose you throw all N beans, one by one, into the bins. What is the probability of throwing the first q beans into bin i, and then the remaining $(N-q)$ beans into other bins? Answer: $\omega_i^q (1-\omega_i)^{N-q}$.

4. Suppose you throw all N beans, one by one, into the bins. How many different ways can you end up with q beans in bin i and $(N-q)$ beans in other bins? This is the same as asking how many ways can one select q beans from among the N total beans. Note that we are not concerned with the order in which the q beans are selected. Answer:
$$\frac{N!}{(N-q)!q!}.$$

5. Suppose you throw all N beans, one by one, into the bins. What is the probability of ending up with q beans in bin i and $(N-q)$ beans in other bins? Answer:
$$\frac{N!}{(N-q)!q!}\omega_i^q (1-\omega_i)^{N-q}.$$

6. Suppose you throw all N beans, one by one, into the bins. What is the probability of throwing the first n_1 beans into bin 1, and the next n_2 beans into bin 2, …, and the final

n_k beans into bin k? Note that the set of numbers $\{n_i\}$ is subject to the restriction that $\sum_{i=1}^{k} n_i = N$. Answer:

$$\omega_1^{n_1}\omega_2^{n_2}\omega_3^{n_3}\ldots\omega_k^{n_k}.$$

7. Suppose you throw all N beans, one by one, into the bins. How many different ways can you end up with n_1 beans in bin 1, n_2 beans in bin 2, …, and n_k beans in bin k? Answer:

$$\frac{N!}{\prod_{i=1}^{k} n_i!}.$$

8. Suppose you throw all N beans, one by one, into the bins. What is the probability of ending up with n_1 beans in bin 1, n_2 beans in bin 2, …, and n_k beans in bin k? Answer:

$$\frac{N!}{\prod_{i=1}^{k} n_i!}\omega_1^{n_1}\omega_2^{n_2}\omega_3^{n_3}\ldots\omega_k^{n_k} = N!\prod_{i=1}^{k}\frac{\omega_i^{n_i}}{n_i!}.$$

Note that the answer to question 8 is what Greiner calls $W_{tot}\{n_i\}$, and thus we have established equation (6.23) on page 148:

$$W_{tot}\{n_i\} = N!\prod_{i=1}^{k}\frac{\omega_i^{n_i}}{n_i!}.$$

The rest of the argument on page 148 is fine, I think. However, at the bottom of the page, he says that you can obtain the Lagrange multiplier λ from the constraint equation $\sum_{i=1}^{k} n_i = N$, but he doesn't do it. It's easy:

$$N = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k}\omega_i e^{\lambda} = e^{\lambda}\left(\sum_{i=1}^{k}\omega_i\right) = e^{\lambda}.$$

So, $\lambda = \ln N$, and we end up with

$$n_i = N\omega_i.$$

But this should be obvious – refer back to question #2 above!

This is the result for the microcanonical ensemble. The result for the canonical ensemble is less obvious; see pages 161-162.