PHYS 705: Classical Mechanics

Euler's Equations

Euler's Equations (set up)

We have seen how to describe the kinematic properties of a rigid body. Now, we would like to get equations of motion for it.

1. We will follow the Lagrangian Formalism that we have developed.

2. For generalized coordinates, we will use the Euler's angles with one point of the rigid body being **fixed** (no translation, just rotation)

3. As we have seen previously, the rotational kinetic energy is given by

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}\omega_i I_{ij}\omega_j$$

4. Choose the body axes to coincide with the Principal axes, then

$$T = \frac{1}{2}I_{ii}\omega_i^2 \qquad \text{(no sum; } I_{ij} \text{ is diagonalized!)}$$

Euler's Equations (set up)

Note:

-We still have the freedom to align $\hat{x}_3\left(\hat{\mathbf{z}}\right)$(from the body axes) to **any** one of the 3 Principal axes.

- The three Euler's angles $\left(\phi,\theta,\psi\right)$ give the orientation of the Principal axes of the body axes relative to the fixed axes.

5. A general rotation (an inf. one here) $d\mathbf{\Omega}$ along a given axis in the body frame can be decomposed into three rotations along the Euler's angles. Similarly, the time rate of change of this rotation $\boldsymbol{\omega} = d\mathbf{\Omega}/dt$ can also be written as,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$$

(we write this as a sum since the angular changes are infinitesimal)

- These three different pieces correspond to the time rate of change of the individual rotations along each of the three Euler's angles.

Euler's Equations (set up)

Now, our task is to project $\boldsymbol{\omega}$ along the three axes in the body coordinate (x_1, x_2, x_3)

We will go through the three individual Euler steps now:

a) $\boldsymbol{\omega}_\phi$: We are in the fixed axes and we do a rotation along the $x_3\,(\hat{\mathbf{z}})$

→ In the fixed axes, we have $(\boldsymbol{\omega}_\phi)_{fixed} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **BCD**

$$(\boldsymbol{\omega}_\phi)_{body} = \mathbf{BCD}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta \\ \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix}$$

Note: Since $(0, 0, \dot{\phi})^T$ is already in the $\hat{\mathbf{z}}$ direction, $\mathbf{D}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}$

Euler's Equations (set up)

b) $\boldsymbol{\omega}_\theta$: This is the (2nd) rotation along the "line of nodes" ($x_1\left(\hat{\mathbf{x}}\right)$ in the intermediate $\left(\xi,\eta,\zeta\right)$ coordinate system)

→ In the intermediate axes, we have $\left(\boldsymbol{\omega}_\theta\right)_\xi = \begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **BC**

$$\left(\boldsymbol{\omega}_\theta\right)_{body} = \mathbf{BC}\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} = \begin{pmatrix}\dot{\theta}\cos\psi\\-\dot{\theta}\sin\psi\\0\end{pmatrix}$$

Note: Since $\left(\dot{\theta},0,0\right)^T$ is already in the $\hat{\mathbf{x}}$ direction, $\mathbf{C}\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} = \begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix}$

Euler's Equations (set up)

c) $\boldsymbol{\omega}_\psi$: Finally, the last rotation is along the $x_3\left(\hat{\mathbf{z}}\right)$ of the $\left(\xi',\eta',\zeta'\right)$

→ In the $\left(\xi',\eta',\zeta'\right)$ axes, we have $\left(\boldsymbol{\omega}_\psi\right)_{\zeta'} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **B**

$$\left(\boldsymbol{\omega}_\psi\right)_{body} = \mathbf{B}\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

Note: Since $\left(0,0,\dot{\psi}\right)^T$ is already in the $\hat{\mathbf{z}}$ direction, $\mathbf{B}\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

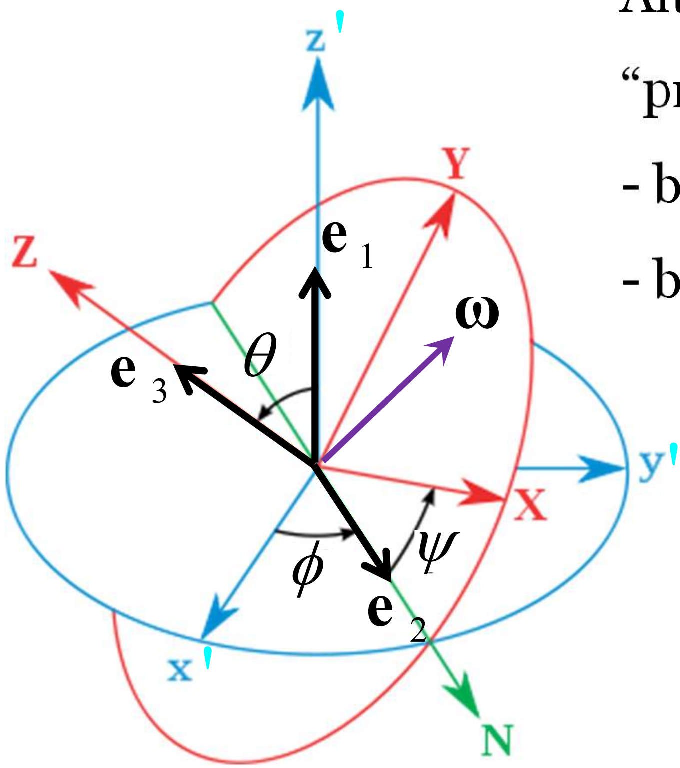
Euler's Equations (set up)

→ Putting all three pieces together, we have

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\phi} + \boldsymbol{\omega}_{\theta} + \boldsymbol{\omega}_{\psi} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

These are the components of $\boldsymbol{\omega}$ expressed in the "body" frame using the Euler's angles.

Euler's Equations (set up)



Alternatively, one can think of how the vector $\boldsymbol{\omega}$ being "projected" along different set of bases:

- basis along the Euler directions $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
- basis along the body frame $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$

$$\boldsymbol{\omega} = \dot{\phi}\,\mathbf{e}_1 + \dot{\theta}\,\mathbf{e}_2 + \dot{\psi}\,\mathbf{e}_3$$

$$\Updownarrow$$

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{x}} + \omega_y \hat{\mathbf{y}} + \omega_z \hat{\mathbf{z}}$$

Note: the basis set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defining an infinitesimal rotation along the Euler angles is NOT an orthogonal set of vectors.

(Note: Here and onward, space frame is primed and body frame is unprimed.)

Euler's Equations (derivation)

Now, we will continue with our equation of motion for a rotating rigid body.

$$T = \frac{1}{2} I_i \omega_i^2$$

I is diagonalized since we've chosen the body axes to lay along the principal axes and we will call the nonzero diagonal elements, $I_{ii} = I_i$

Without further assuming the nature of the applied forces acting on this system, we will use the following general form of the E-L equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

Q_i is the generalized force (including forces derivable from conservative and non-conservative sources)

Euler's Equations (derivation)

Let calculate the equation of motion explicitly for ψ :

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = \left(I_i\omega_i\right)\frac{\partial \omega_i}{\partial \dot{\psi}} \qquad \text{(E's sum)}$$

$$= \left(I_1\omega_1\right)\frac{\partial \omega_1}{\partial \dot{\psi}} + \left(I_2\omega_2\right)\frac{\partial \omega_2}{\partial \dot{\psi}} + \left(I_3\omega_3\right)\frac{\partial \omega_3}{\partial \dot{\psi}}$$

$$= \left(I_1\omega_1\right)\left(0\right) + \left(I_2\omega_2\right)\left(0\right) + \left(I_3\omega_3\right)\left(1\right)$$

$$= I_3\omega_3$$

$$T = \frac{1}{2}I_i\omega_i^2$$

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_3\dot{\omega}_3$$

Euler's Equations (derivation)

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

Now, we need

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} = I_1\omega_1\frac{\partial \omega_1}{\partial \psi} + I_2\omega_2\frac{\partial \omega_2}{\partial \psi} + I_3\omega_3\frac{\partial \omega_3}{\partial \psi}$$

Note that:

$$\frac{\partial \omega_1}{\partial \psi} = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi = \omega_2$$

$$\frac{\partial \omega_3}{\partial \psi} = 0$$

$$\frac{\partial \omega_2}{\partial \psi} = -\dot{\phi}\sin\psi\sin\theta - \dot{\theta}\cos\psi = -\omega_1$$

Thus, we have,

$$\frac{\partial T}{\partial \psi} = I_1\omega_1\left(\omega_2\right) + I_2\omega_2\left(-\omega_1\right) + 0 = I_1\omega_1\omega_2 - I_2\omega_2\omega_1$$

Euler's Equations (derivation)

Now, we need to calculate the generalized force with respect to ψ :

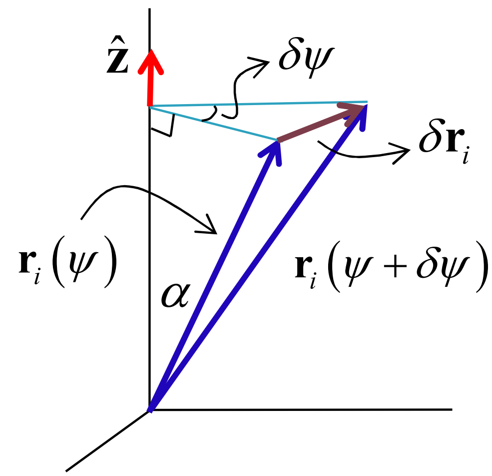
Since the Euler angle ψ is associated with a rotation about the $\hat{\mathbf{z}}$ axis in the "body" frame, we have,

$$Q_\psi \equiv \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \psi} = \sum_i \mathbf{F}_i \cdot \left(\hat{\mathbf{z}} \times \mathbf{r}_i \right)$$

$$= \hat{\mathbf{z}} \cdot \mathbf{N} = N_3$$

used

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$



$$|\delta \mathbf{r}_i| = |\mathbf{r}_i| \sin\alpha \, \delta\psi$$

$$\frac{\partial \mathbf{r}_i}{\partial \psi} = \hat{\mathbf{z}} \times \mathbf{r}_i$$

Euler's Equations (derivation)

Finally, putting everything together, the E-L equation gives,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_\psi$$

$$\Rightarrow \quad I_3\dot{\omega}_3 - I_1\omega_1\omega_2 + I_2\omega_1\omega_2 = N_3 \quad \Rightarrow \quad \boxed{I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2 = N_3}$$

- One can calculate the E-L equation for θ, ϕ but (they are ugly) we are not doing them here !

- There is a smarter way to get EOM for the other two dofs…

→ Since nothing required our choice of ω_3 to lay along $\hat{\mathbf{z}}(x_3)$. Then, by a symmetry argument, the other components of $\dot{\boldsymbol{\omega}}$ should have a SIMILAR form.

Euler's Equations (derivation)

This then gives,

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 &= N_3 \end{aligned}}$$

(same cyclic symmetry as the equation for ω_3)

In principle, one can get out the $\dot{\omega}_2$ *and* $\dot{\omega}_3$ equation by solving for $\dot{\omega}_2$ *and* $\dot{\omega}_3$ simultaneously from the θ, ϕ Euler-Lagrange equations.

These are called the Euler's Equations and the motion is described in terms of the Principal Moments !

Euler's Equations (derivation)

The following is Goldstein's (Newtonian) derivation. We start with,

$$\mathbf{N} = \left(\frac{d\mathbf{L}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

Writing this vector equation out in the components of the *body* axes,

$$N_i = \frac{dL_i}{dt} + \varepsilon_{ijk}\omega_j L_k$$

Choose the body axes to coincide with the Principal axes, so that

$$L_i = I_i \omega_i$$ (no sum, just writing out the components)

$$N_i = I_i\left(\frac{d\omega_i}{dt}\right) + \varepsilon_{ijk}\omega_j I_k \omega_k$$ (no sum)

Euler's Equations (derivation)

Writing this index equation out explicitly for $i=1,2,3$, we have,

$$I_1\dot{\omega}_1+\varepsilon_{123}\omega_2 I_3\omega_3+\varepsilon_{132}\omega_3 I_2\omega_2=\boxed{I_1\dot{\omega}_1-\left(I_2-I_3\right)\omega_2\omega_3=N_1}$$

$$I_2\dot{\omega}_2+\varepsilon_{231}\omega_3 I_1\omega_1+\varepsilon_{213}\omega_1 I_3\omega_3=\boxed{I_2\dot{\omega}_2-\left(I_3-I_1\right)\omega_3\omega_1=N_2}$$

$$I_3\dot{\omega}_3+\varepsilon_{312}\omega_1 I_2\omega_2+\varepsilon_{321}\omega_2 I_1\omega_1=\boxed{I_3\dot{\omega}_3-\left(I_1-I_2\right)\omega_1\omega_2=N_3}$$

So, this gives us the same set of Euler's equations as previously.

The Euler's Equations describes motion in the body frame. $\boldsymbol{\omega}$ and $\mathbf{N}$ are vectors expressed in the body frame.

Torque Free Motion of a Symmetric Top

A symmetric top means that: $I_1 = I_2 \neq I_3$

For concreteness, let $I_1 = I_2 > I_3$ (example will be a long cigar-like objects such as a juggling pin)

Euler equations (torque free) are:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\,\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\,\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\,\omega_1\omega_2 = 0 \end{aligned}$$

or

$$\left(\frac{d\mathbf{L}}{dt}\right)_{body} = -\boldsymbol{\omega}\times\mathbf{L}$$

$$\dot{\mathbf{L}} = \mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\times\mathbf{L}$$

Trivial case ($\boldsymbol{\omega}$ is along one of the principal axes):

⇨ $\boldsymbol{\omega}$ is along one of the eigendirection of $\mathbf{I}$ and $\mathbf{L} \,||\, \boldsymbol{\omega}$

$\dot{\mathbf{L}} = \mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\times\mathbf{L} = \mathbf{0}$ ⇨ $\dot{\boldsymbol{\omega}} = 0$

Torque Free Motion of a Symmetric Top

Interesting case ($\boldsymbol{\omega}$ is NOT along one of the principal axes):

We still have, $\dot{\omega}_3 = 0$ since $I_1 = I_2$

$$\boxed{\omega_3 = const}$$

Note: $\hat{x}_3$ is along the body's symmetry axis (symmetric top).

And, the rest of the Euler equations give,

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_2\omega_3$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1$$

Note: $\boxed{I_1 = I_2}$

Torque Free Motion of a Symmetric Top

Let $\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = const$

Then, the remaining two Euler's equations reduce simply to,

$$\dot{\omega}_1 = -\Omega\omega_2$$
$$\dot{\omega}_2 = \Omega\omega_1$$

Taking the derivative of the top equation and substitute the bottom on the right, we have,

$$\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\Omega\left(\Omega\omega_1\right) = -\Omega^2\omega_1$$

Since, $\Omega^2 \geq 0$ we have the solution:

A, φ_0 will be determined by ICs

$$\boxed{\omega_1(t) = A\cos\left(\Omega t + \varphi_0\right) \qquad \text{and} \qquad \omega_2(t) = A\sin\left(\Omega t + \varphi_0\right)}$$

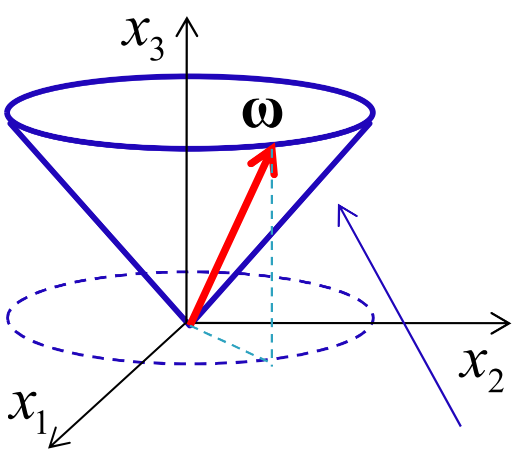
Torque Free Motion of a Symmetric Top

Looking at this deeper… First in the "body" frame,

- We know that ω_3 is a constant and ω_1 & ω_2 oscillates harmonically in a circle.

So, $\omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2 = const$ ⇨ $|\boldsymbol{\omega}| = const$

⇨ In the "body" axes, this description for $\boldsymbol{\omega}$ can be visualized as $\boldsymbol{\omega}$ precessing about $\hat{x}_3$.



-The projection of $\boldsymbol{\omega}$ onto the x_3 axis is fixed.

-The projection of $\boldsymbol{\omega}$ onto the $x_1 - x_2$ plane rotates as a parametric circle with a rate of

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = const$$

(This is called the "body" cone)

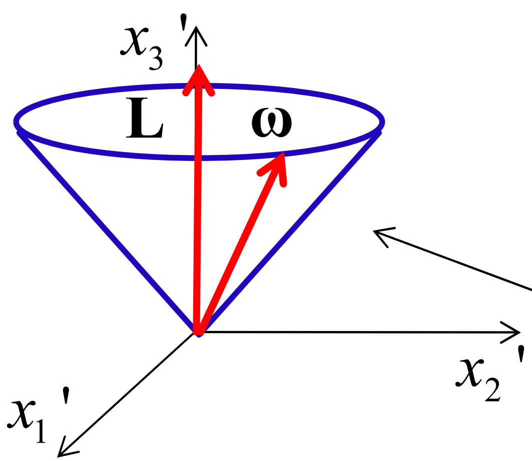
Torque Free Motion of a Symmetric Top

Now, let look at this same situation in the "fixed" frame,

Observations:

1. Energy is conserved in this problem so that $T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = const$
2. The situation is torque free so that **L** is fixed in space.

⇨ #1 and #2 means that $\boldsymbol{\omega}$ must also precesses around **L** in the fixed frame



- Assume **L** lies along the x_3' axis in the fixed frame.
- **L** is a constant vector
- The dot product $\boldsymbol{\omega}\cdot\mathbf{L}$ must remain constant.

(This is called the "space" cone)

Torque Free Motion of a Symmetric Top

Observations (in the fixed axes) cont:

3. The three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3(body)$ always lie on a plane.

Consider the following product:

$$\mathbf{L}\cdot\left(\boldsymbol{\omega}\times\hat{\mathbf{x}}_3\right) \qquad \text{where } \hat{\mathbf{x}}_3 \text{ is in the } \hat{\mathbf{z}} \text{ direction in the body axes}$$

$$=\mathbf{L}\cdot\left(\omega_2\hat{\mathbf{x}}_1-\omega_1\hat{\mathbf{x}}_2\right)=\omega_2\left(\mathbf{L}\cdot\hat{\mathbf{x}}_1\right)-\omega_1\left(\mathbf{L}\cdot\hat{\mathbf{x}}_2\right)$$

Since the body axes are chosen to lie along the principal axes, we have

$$L_i = I_i\omega_i \ (no\ sum)$$

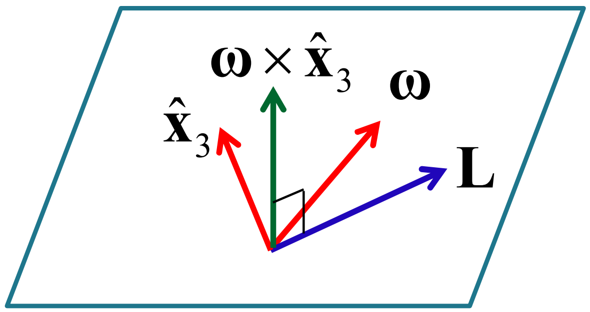
$$\Rightarrow \quad \mathbf{L}\cdot\left(\boldsymbol{\omega}\times\hat{\mathbf{x}}_3\right)=\omega_2\left(I_1\omega_1\right)-\omega_1\left(I_2\omega_2\right)=0$$

(for a symmetric top) $I_1 = I_2$

Torque Free Motion of a Symmetric Top

Observations (in the fixed axes) cont:

This means that all three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane.



$$\mathbf{L} \cdot \left(\boldsymbol{\omega} \times \hat{\mathbf{x}}_3 \right) = 0$$

(for a symmetric top with or without torque)

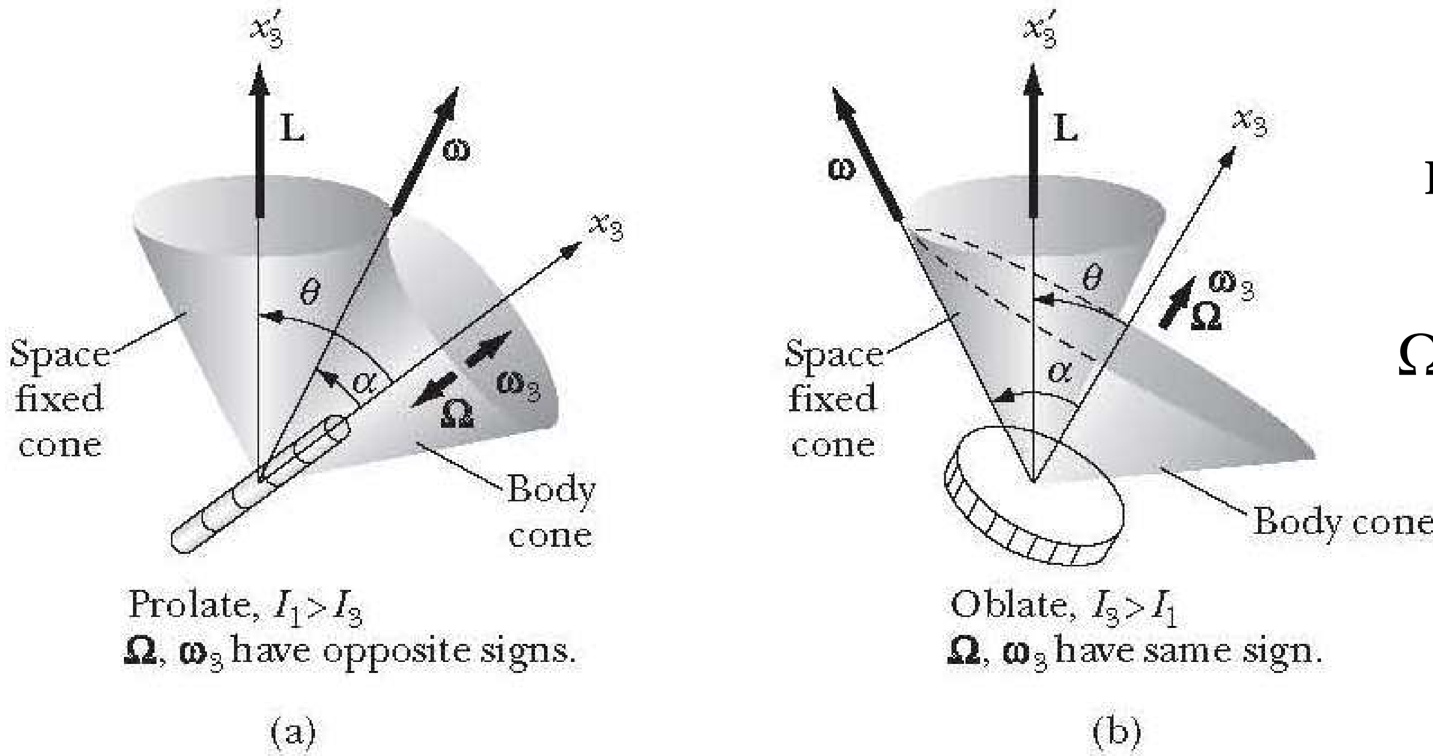
http://demonstrations.wolfram.com/AngularMomentumOfARotatingRigidBody/

Summary:

- $\boldsymbol{\omega}$ precesses around the "body" cone
- $\boldsymbol{\omega}$ also precesses around the "space" cone
- All three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane
- $\mathbf{L}$ is chosen to align with $\hat{\mathbf{x}}_3\,'$ in the space axes

Torque Free Motion of a Symmetric Top

This can be visualized as the body cone rolling either inside or outside of the space cone !



Precession Rate

$$\Omega = \left(\frac{I_3 - I_1}{I_1} \right) \omega_3$$

Case 1: $I_1 > I_3$

Case 2: $I_1 < I_3$

Stability of General Torque Free Motion

Consider torque-free motion for a rigid body with $I_1 > I_2 > I_3$

Again, we have chosen the body axes to align with the principal axes.

As an example, we will consider rotation near the x_1 axis (similar analysis can be done near the other two principal axes).

→ this means that we have,

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{x}}_1 + \lambda(t)\,\hat{\mathbf{x}}_2 + \mu(t)\,\hat{\mathbf{x}}_3$$

where $\lambda(t), \mu(t)$ are *small* time-dependent perturbation to the motion

For stability analysis, we wish to analyze the time evolution of these two quantities to see if they remain small or will they blow up.

Stability of General Torque Free Motion

Plugging our perturbation into the Euler's equations, we have

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \quad \rightarrow \quad I_1\dot{\omega}_1 - (I_2 - I_3)\lambda\mu = 0$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0 \quad \rightarrow \quad I_2\dot{\lambda} - (I_3 - I_1)\mu\omega_1 = 0$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \quad \rightarrow \quad I_3\dot{\mu} - (I_1 - I_2)\omega_1\lambda = 0$$

Assume small perturbations and drops higher order terms ($\lambda\mu$), the first equation gives,

$$\dot{\omega}_1 = 0 \quad \Rightarrow \quad \omega_1 = const$$

Stability of General Torque Free Motion

And, the other two equations reduces to,

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \mu = 0$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda = 0$$

Taking the derivative of the top equation and substitute the bottom into it,

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \dot{\mu} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda$$

$$\ddot{\lambda} = \left(\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \omega_1^{\ 2} \right) \lambda$$

Stability of General Torque Free Motion

Since we have chosen to have $I_1 > I_2 > I_3$, the constant

$$\Omega^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^{\ 2} > 0$$

(Note: we have switch the order of I_1, I_3 so that Ω^2 is explicitly positive.)

And, we can write

$$\ddot{\lambda} = -\Omega^2 \lambda$$

The solution to this ODE is oscillatory, i.e.,

$$\lambda(t) = Ae^{i\Omega t} + Be^{-i\Omega t}$$

and $$\mu(t) = A'e^{i\Omega t} + B'e^{-i\Omega t}$$ where *A, B, A', & B'* depends on ICs

Thus, both of the small perturbations are oscillatory and the rotation about the x_1 axis is stable !

Stability of General Torque Free Motion

With a similar calculation for rotation near the x_3 , one can show again that small perturbations are oscillatory and motion about the x_3 axis is stable.

However, a similar analysis will show that the oscillatory motion for the perturbations will become exponential if we consider rotation near the x_2 axis.

Summary:

Without any applied torque, motion around the principle axes with the largest and the smallest principal moments are stable while motion around the intermediate axis is unstable.

http://www.youtube.com/watch?v=XALe27bnUm8