PHYS 705: Classical Mechanics

Non-inertial Reference Frames
Vectors in Rotating Frames

Infinitesimal Transformations

From our previous discussion, we have established that…

→ any orientation of a rigid body (with one point fixed) can be represented by a single rotation about a particular axis.

→ And this rotation can be parameterized by three parameters (3 Euler angles)

Question: Can we represent these three parameters by a vector?

Suppose the two rotations Op_A and Op_B are represented by two such "vectors",

Then, for them to be actual vectors, the two operations must commute, i.e.

$$Op_A \oplus Op_B = Op_B \oplus Op_A$$

However, from our previous discussion, *finite* rotations are not commutative as an operation.

Infinitesimal Transformations

Recall that if we represent the rotation $Op_{A,B}$ by matrices, **A,B** , then the "addition" of two rotations Op_B and Op_B will be given by a matrix multiplication, i.e.,

$$Op_A \oplus Op_B \quad \rightarrow \quad \mathbf{A} \cdot \mathbf{B}$$

And, matrix multiplications are not commutative : $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Thus, **finite** rotations CANNOT be represented as vectors !

However, as we will demonstrate next....

Infinitesimal rotations can be represented by vectors !

Infinitesimal Transformations

Let consider a general infinitesimal transformation given by the following,

$$x'_i = x_i + \varepsilon_{ij} x_j = \left(\delta_{ij} + \varepsilon_{ij}\right) x_j$$

so that the unprimed axes x_i is changed to x'_i by an infinitesimal operator ε_{ij}.

In matrix notations, we have, $\mathbf{x}' = \left(\mathbf{I} + \boldsymbol{\varepsilon}\right)\mathbf{x}$

Since $\boldsymbol{\varepsilon}$ is assumed to be small, we will show that the order of infinitesimal transformation is *not* important.

Infinitesimal Transformations

- Let say we have two infinitesimal transformations: $(\mathbf{I}+\boldsymbol{\varepsilon}_1)$, $(\mathbf{I}+\boldsymbol{\varepsilon}_2)$

$$Op_1 \oplus Op_2 \quad \Rightarrow \quad (\mathbf{I}+\boldsymbol{\varepsilon}_1)(\mathbf{I}+\boldsymbol{\varepsilon}_2) = \mathbf{I}^2 + \boldsymbol{\varepsilon}_1\mathbf{I} + \mathbf{I}\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 = \mathbf{I} + \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + O\left(\varepsilon^2\right)$$

- Switching the order of the operations,

$$Op_2 \oplus Op_1 \quad \Rightarrow \quad (\mathbf{I}+\boldsymbol{\varepsilon}_2)(\mathbf{I}+\boldsymbol{\varepsilon}_1) = \mathbf{I}^2 + \boldsymbol{\varepsilon}_2\mathbf{I} + \mathbf{I}\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_1 = \mathbf{I} + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_1 + O\left(\varepsilon^2\right)$$

- By *neglecting higher order terms*, these two expressions are the **same** since matrix addition is commutative.

→Infinitesimal transformation is commutative.

Infinitesimal Transformations

- Now, let consider an *inverse* of an infinitesimal transformation, $\mathbf{A} = \mathbf{I} + \boldsymbol{\varepsilon}$

It is given simply by: $\mathbf{A}^{-1} = \mathbf{I} - \boldsymbol{\varepsilon}$

CHECK:

$$\mathbf{A}\mathbf{A}^{-1} = (\mathbf{I} + \boldsymbol{\varepsilon})(\mathbf{I} - \boldsymbol{\varepsilon}) = \mathbf{I} \quad \text{(to first order)}$$

- Then, let see what property will $\boldsymbol{\varepsilon}$ have if we want this infinitesimal transformation to be *orthogonal*.

- For *orthogonal* transformations, we need to have $\mathbf{A}^{T} = \mathbf{A}^{-1}$

- This then gives, $\mathbf{A}^{T} = \mathbf{I} + \boldsymbol{\varepsilon}^{T} = \mathbf{A}^{-1} = \mathbf{I} - \boldsymbol{\varepsilon}$ ⇨ $\boldsymbol{\varepsilon}^{T} = -\boldsymbol{\varepsilon}$

⇨ $\boldsymbol{\varepsilon}$ is an anti-symmetric matrix

Infinitesimal Transformations

- Further note that,

$$\det \mathbf{A} = \det\left(\mathbf{I} + \boldsymbol{\varepsilon}\right) = +1 \quad \text{(to first order)}$$

An infinitesimal *orthogonal* transformation corresponds to a proper rotation.

- Since $\boldsymbol{\varepsilon}$ is anti-symmetric, we can write it generally in component form as,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix} \qquad \left(d\Omega_1, d\Omega_2, d\Omega_3\right)$$

- So, $\boldsymbol{\varepsilon}$ has only *three* independent components: $\left(d\Omega_1, d\Omega_2, d\Omega_3\right)$
- Any infinitesimal rotation can be specified by these three parameters

Infinitesimal Rotations

- As an example, we can explicitly write out the *infinitesimal* Euler rotation,

With $\theta \cong d\theta, \phi \cong d\phi, \psi \cong d\psi$

$$\mathbf{A} = \begin{pmatrix} \cos d\psi \cos d\phi - \cos d\theta \sin d\phi \sin d\psi & \cos d\psi \sin d\phi + \cos d\theta \cos d\phi \sin d\psi & \sin d\theta \sin d\psi \\ -\sin d\psi \cos d\phi - \cos d\theta \sin d\phi \cos d\psi & -\sin d\psi \sin d\phi + \cos d\theta \cos d\phi \cos d\psi & \sin d\theta \cos d\psi \\ \sin d\theta \sin d\phi & -\sin d\theta \cos d\phi & \cos d\theta \end{pmatrix}$$

Then, with

$$\cos d\theta = \cos d\phi = \cos d\psi \cong 1$$
$$\sin d\theta \cong d\theta$$
$$\sin d\phi \cong d\phi$$
$$\sin d\psi \cong d\psi$$

gives…

Infinitesimal Rotations

- As an example, we can explicitly write out the *infinitesimal* Euler rotation,

$$\mathbf{A} = \begin{bmatrix} 1 & (d\phi + d\psi) & 0 \\ -(d\phi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{bmatrix} = \mathbf{I} + \begin{bmatrix} 0 & d\phi + d\psi & 0 \\ -d\phi - d\psi & 0 & d\theta \\ 0 & -d\theta & 0 \end{bmatrix} = \mathbf{I} + \boldsymbol{\varepsilon}$$

$\sin dx \simeq dx,$
$\cos dx \simeq 1,$
and keeping
only linear terms

Now, for the three infinitesimal Euler parameters,

we formally define,

$$d\mathbf{\Omega} \equiv \left(d\theta, 0, (d\phi + d\psi)\right)^T$$

- For notational purpose, we write $d\mathbf{\Omega}$ as a differential but it is NOT an actual differential of a vector.
- But $d\mathbf{\Omega}$ stands for a vector of differential changes associated with the infinitesimal transformation matrix $\boldsymbol{\varepsilon}$

Infinitesimal Rotations

- We will now show that these three quantities $\left(d\Omega_1, d\Omega_2, d\Omega_3\right)$ form a particular kind of vector.

- The change $d\mathbf{r}$ in the coordinates $\mathbf{r}$ under this infinitesimal rotation can be expressed as,

$$d\mathbf{r} \equiv \mathbf{r}' - \mathbf{r} = \left(\mathbf{I} + \boldsymbol{\varepsilon}\right)\mathbf{r} - \mathbf{r} = \boldsymbol{\varepsilon}\mathbf{r}$$

- Writing this out explicitly in components, we have,

$$d\mathbf{r} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \boldsymbol{\varepsilon}\mathbf{r} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{aligned} dx_1 &= x_2 d\Omega_3 - x_3 d\Omega_2 \\ dx_2 &= x_3 d\Omega_1 - x_1 d\Omega_3 \\ dx_3 &= x_1 d\Omega_2 - x_2 d\Omega_1 \end{aligned}$$

- The result looks like the cross product between two vectors:

$$d\mathbf{r} = \mathbf{r} \times d\boldsymbol{\Omega} \qquad \text{with} \qquad \mathbf{r} = \left(x_1, x_2, x_3\right)^T \qquad d\boldsymbol{\Omega} = \left(d\Omega_1, d\Omega_2, d\Omega_3\right)^T$$

$d\Omega$ as a Pseudovector

- Under an orthogonal transformation **B**, the anti-symmetric matrix $\boldsymbol{\varepsilon}$ transforms according to the similarity transform:

$$\boldsymbol{\varepsilon}' = \mathbf{B}^{-1}\,\boldsymbol{\varepsilon}\,\mathbf{B}$$

- Recall that a regular vector (called **polar vector**) **r**, it must satisfy the transformation rule:

$$x'_i = b_{ij}x_j \qquad \text{where } b_{ij} \text{ is orthogonal, i.e, } b_{ij}b_{ik} = \delta_{jk}$$

- One can also show that $d\boldsymbol{\Omega}$ must transform as,

$$d\Omega'_i = \det(\mathbf{B})\, b_{ij}\, d\Omega_j \qquad (*) \qquad \text{(not shown here)}$$

So, if $\det(\mathbf{B}) = 1$ $d\Omega_i$, transforms like a regular vector but if $\det(\mathbf{B}) \neq 1$, it is not.

$d\Omega$ as a Pseudovector

- Under an orthogonal transformation **B**, the anti-symmetric matrix $\boldsymbol{\varepsilon}$ transforms according to the similarity transform:

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- One can also show that $d\boldsymbol{\Omega}$ must transform as,

$$d\Omega'_i = \det(\mathbf{B})\, b_{ij}\, d\Omega_j \qquad (*) \qquad \text{(not shown here)}$$

- Vectors transforming according to (*) is called an **axial vector** (pseudovector)

$d\Omega$ as a Pseudovector

- Also, recall that if **B** is an orthogonal transformation,

$$\det \mathbf{B} = \pm 1$$

→ det **B** = + 1, then **B** is called proper, (rotation only, no inversion)

→ det **B** = -1, then **B** is called improper (with inversion)

- So, we have the following distinction between a polar and axial vector under an *improper* orthogonal transformation **B**:

$$\mathbf{v}' = \mathbf{B}\mathbf{v} \qquad \text{if } \mathbf{v} \text{ is a polar vector}$$

$$\mathbf{v}' = -\mathbf{B}\mathbf{v} \qquad \text{if } \mathbf{v} \text{ is an axial vector (or pseudovector)}$$

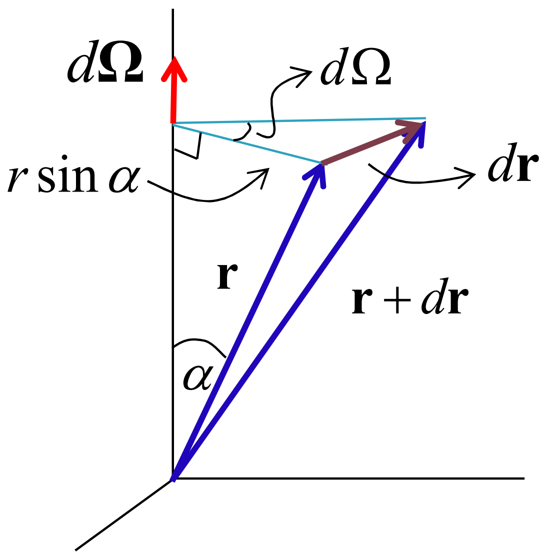
- Common examples of axial vectors in physics involves the cross product of two polar vectors such as angular velocity, angular momentum,…

Geometric View of an Infinitesimal Rotations

We just saw that an inf. rotation $d\mathbf{\Omega}$ leads to an inf. change $d\mathbf{r}$ for $\mathbf{r}$:

$$d\mathbf{r} = \mathbf{r} \times d\mathbf{\Omega}$$ (**This is in the *passive* view)

Here is the same situation in the *active* view:



Here, we are rotating the vector $\mathbf{r}$ (active view) instead of rotating the coordinates (passive view) → thus the reverse order of the cross prod.

magnitude: $r\sin\alpha\, d\Omega$

direction: *RHR*

⇨ $$d\mathbf{r} = d\mathbf{\Omega} \times \mathbf{r}$$

(We will take this active view from now on.)

Geometric View of an Infinitesimal Rotations

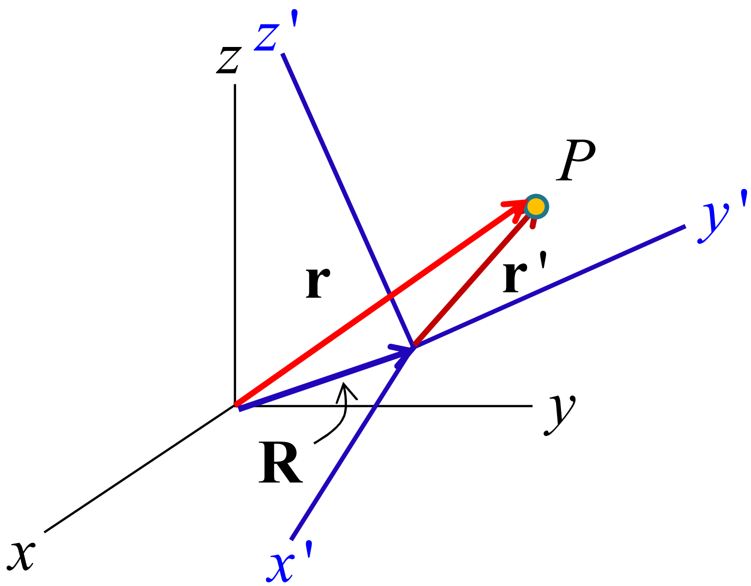
Dividing *dt* from both sides of the equation, we have,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}$$ (the familiar definition for angular velocity)

where $\boldsymbol{\omega} = \frac{d\mathbf{\Omega}}{dt}$ is the instantaneous angular velocity

where $\boldsymbol{\omega}$ lies along the axis of infinitesimal rotation in the time interval $\left[t, t+dt\right]$ in a direction known as the **instantaneous axis of rotation**.

Rate of Change of a Vector under Rotation



Looking at infinitesimal rotation more generally:

We are interested in describing vectors in the "body" (primed) frame as measured in the "fixed" (unprimed) frame when the rigid body frame is being (infinitesimally) rotated about a fixed axis.

($\mathbf{R}$ is assumed to be fixed for now.)

- For simplicity, we also assume (for now) that a point *P* described by $\mathbf{r}'$ in the "body" frame does *not* move wrt the body frame.

Note: Our discussion can be for any vector $\mathbf{G}$ instead of the specific position vector $\mathbf{r}'$ in the body frame.

Rate of Change of a Vector in under Rotation

- However, as viewed from the "fixed" frame, $\mathbf{r}'$ will be moving (due to $d\boldsymbol{\Omega}$)

- From our discussion on rotation before, viewed in the fixed frame,

$$\left(d\mathbf{r}'\right)_{fixed} = d\boldsymbol{\Omega} \times \mathbf{r}'$$

- Or, dividing by dt, we have,

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} = \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}' = \boldsymbol{\omega} \times \mathbf{r}'$$

- More generally, if we also now allow $\mathbf{r}'$ to move wrt the "body" frame as well,

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} = \left(\frac{d\mathbf{r}'}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}'$$

Rate of Change of a Vector in under Rotation

- Although our discussion was for a given position vector $\mathbf{r}'$ in the "body" frame, the discussion applies equally well to ANY vector $\mathbf{G}$ in the "body" frame.

$$\left(\frac{d\mathbf{G}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{G}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{G}$$

- So, it is useful to abstract out this operation for any vectors,

$$\boxed{\left(\frac{d}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times}$$

Rate of Change of a Vector in under Rotation

This relation can also be formally derived.

The components of a vector $\mathbf{G}$ measured in the body frame and in the fixed frame are related by an orthogonal transformation

$$\text{or} \quad \begin{cases} \mathbf{G}(body) = \mathbf{AG}(fixed) \\ G'_i = a_{ij} G_j \end{cases}$$

where $\mathbf{A}$ or a_{ij} is the Euler rotation matrix

Reversing the expression between the fixed (unprimed) and body (primed) frame,

$$G_i = \left(a^{-1}\right)_{ij} G'_j = \left(a^T\right)_{ij} G'_j = a_{ji} G'_j$$

Rate of Change of a Vector in under Rotation

As the body moves infinitesimally in time, the components of G'_i in the body frame will change by dG'_j and the instantaneous rotation matrix will be given by the infinitesimal rotation matrix,

$$\mathbf{A} = \mathbf{I} + \boldsymbol{\varepsilon}\mathbf{A} = \mathbf{I} + \boldsymbol{\varepsilon} \qquad a_{ij} = \delta_{ij} + \varepsilon_{ij}$$

So, the components in the fixed frame is then changed by ,

$$G_i + dG_i = \overbrace{\left(\delta_{ji} + \varepsilon_{ji}\right)}^{\mathbf{A}^T\left(a_{ji}\right)}\left(G'_j + dG'_j\right)$$

Keeping terms up to to 1st order, we have,

$$G_i + dG_i = G'_i + dG'_i + \boxed{\varepsilon_{ji} G'_j}$$

Note, the order of the indices is not the same as the regular matrix product

Rate of Change of a Vector in under Rotation

Without loss of generality, we pick our fixed frame to be instantaneously coincident with the body frame at time *t* so that we choose

$$G_i = G'_i \quad \text{at time t}$$

BUT, as one can see from the last equation in the previous page, the differential of **G** in the body and fixed frame are in general *not* the same and is related by,

$$dG_i = dG'_i + \varepsilon_{ji} G'_j$$

Using the anti-symmetric property of $\boldsymbol{\varepsilon}$, i.e., $\varepsilon_{ji} = -\varepsilon_{ij}$

$$dG_i = dG'_i - \varepsilon_{ij} G'_j$$

Rate of Change of a Vector in under Rotation

$$dG_i = dG'_i - \varepsilon_{ij} G'_j$$

Then, recall that we have $\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$

And using the Levi Civita tensor $\varepsilon_{ijk} = \begin{cases} +1, & ijk \text{ is of forward permutation} \\ -1, & ijk \text{ is of backward permutation} \\ 0, & \text{otherwise} \end{cases}$

The we can write $\boldsymbol{\varepsilon}$ in terms of ε_{ijk},

NOTE $\boldsymbol{\varepsilon}$ or ε_{ij} is not ε_{ijk}

$$\varepsilon_{ij} = \varepsilon_{ijk} d\Omega_k = -\varepsilon_{ikj} d\Omega_k$$

Rate of Change of a Vector in under Rotation

So, we have, $dG_i = dG'_i + \varepsilon_{ikj} d\Omega_k G'_j$

or $d\mathbf{G}_{fixed} = d\mathbf{G}_{body} + d\mathbf{\Omega} \times \mathbf{G}$

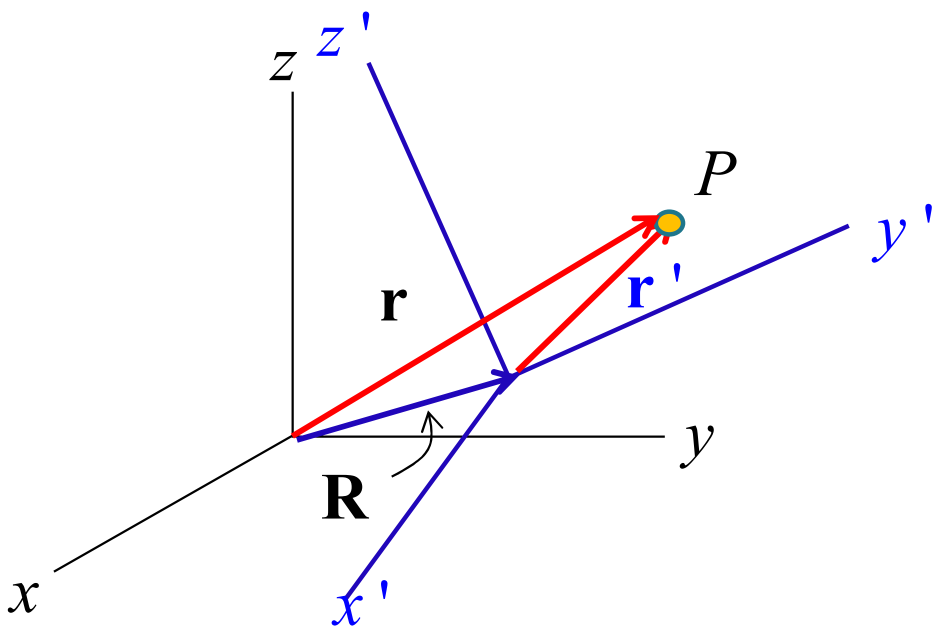
And, finally $\left(\dfrac{d\mathbf{G}}{dt}\right)_{fixed} = \left(\dfrac{d\mathbf{G}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{G}$

And the operator equation again,

$$\left(\frac{d}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times$$

Acceleration in a Rotating Frame

- Let use the "operator" equation to calculate the velocity and acceleration of a point P in the body frame as measured in the **fixed frame**



unprimed → wrt fixed frame

primed → wrt body frame

$$\mathbf{r} = \mathbf{R} + \mathbf{r}'$$

(Here, we also allow **R** to move as well.)

Acceleration in a Rotating Frame

- Consider a change of the position vector **r** wrt the fixed frame, we have,

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{R}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{r}'}{dt}\right)_{fixed}$$

- ***r'*** is in the rotating *body* frame, $(d\mathbf{r}'/dt)$ wrt the fixed frame will have two pieces,

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} = \left(\frac{d\mathbf{r}'}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}'$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{R}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{r}'}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}'$$

vel of *P* rel to fixed axes

vel of rotating frame origin

vel of *P* rel to rotating frame

extra piece due to rotation of the body frame

$$\Rightarrow \quad \mathbf{v}_f = \mathbf{v}_R + \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}'$$

Acceleration in a Rotating Frame

- Differentiate in the *fixed* frame again, we have

$$\left(\frac{d\mathbf{v}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{v}_R}{dt}\right)_{fixed} + \left(\frac{d\mathbf{v}'}{dt}\right)_{fixed} + \boldsymbol{\omega}\times\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} + \dot{\boldsymbol{\omega}}\times\mathbf{r}'$$

Both **v**’ and **r**’ are vectors in the “body” frame but the differentiation is measured in the fixed frame so that we need to apply the operator equation again to expand them.

Acceleration in a Rotating Frame

$$\left(\frac{d\mathbf{v}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{v}_R}{dt}\right)_{fixed} + \left(\frac{d\mathbf{v}'}{dt}\right)_{fixed} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} + \dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

$$\left(\frac{d\mathbf{v}}{dt}\right)_{f} = \left(\frac{d\mathbf{v}_R}{dt}\right)_{f} + \left[\left(\frac{d\mathbf{v}'}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{v}'\right] + \boldsymbol{\omega} \times \left[\left(\frac{d\mathbf{r}'}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}'\right] + \dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

$$\Rightarrow \quad \mathbf{a}_f = \mathbf{a}_R + \mathbf{a}_r + 2(\boldsymbol{\omega} \times \mathbf{v}') + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

acc of P rel to fixed axes

acc of rotating frame origin

acc of P rel to rotating frame

vel of P in the rotating frame

pos of P in the rotating frame

Acceleration in a Rotating Frame

- To gain some physical insights into the meanings of the various terms,

1. We look at the situation where the "body "frame is rotating with a constant angular velocity around a fixed axis and the origin of the "body" frame is not moving, i.e.,

$$\dot{\boldsymbol{\omega}} = 0 \quad \text{and} \quad \mathbf{a}_R = 0$$

2. Newton 2nd Law applies in *inertia* frames only. That is why we have calculated $\mathbf{a}_f$ explicitly in the fixed frame and we have,

$$\mathbf{F} = m\mathbf{a}_f$$

$$\mathbf{F} = m\mathbf{a}_r + 2m(\boldsymbol{\omega}\times\mathbf{v}') + m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')$$

where **F** is the net force acting on *m* as observed in the "fixed" frame.

Acceleration in a Rotating Frame

- Rearranging the terms, we can rewrite the equation into the following form,

$$\mathbf{F} - 2m(\boldsymbol{\omega}\times\mathbf{v}') - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') = m\mathbf{a}_r$$

- Defining $\boxed{\mathbf{F}_{eff} = \mathbf{F} - 2m(\boldsymbol{\omega}\times\mathbf{v}') - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')}$

- We can re-interpret the Newton 2nd law as an equation applying in the "rotating" frame,

$$\boxed{\mathbf{F}_{eff} = m\mathbf{a}_r}$$

→ This is "**F** = m**a**" in the "rotating" frame !

Acceleration in a Rotating Frame

$$\mathbf{F}_{eff} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}') - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

Notes:

- **F** is the actual physical forces in the inertia frame
- The two extra terms corresponding to "apparent" forces (a penalty you pay) for NOT being in an inertial frame.

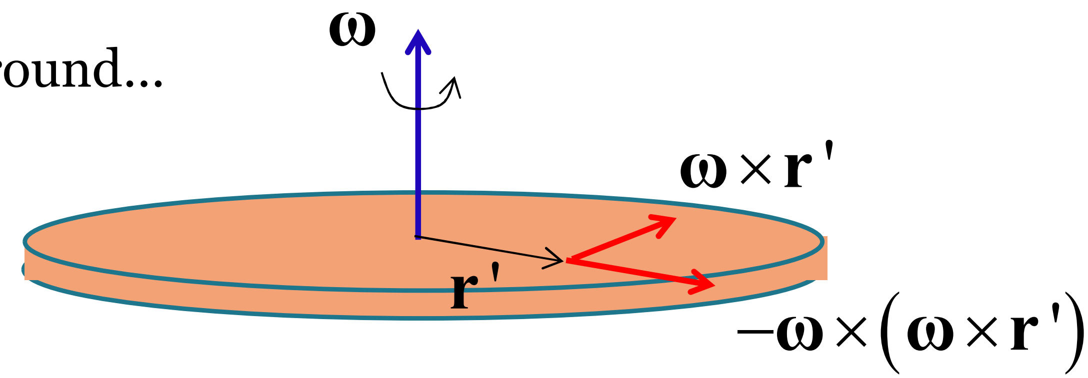
$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ → The familiar "centrifugal" force

$-2m(\boldsymbol{\omega} \times \mathbf{v}')$ → The coriolis force

- These two fictitious forces are not associated with any real forces acting on objects in an inertial frame and they are **not** visible to an inertial observer !

Example of Fictitious Forces in a Rotating Frame

Just *standing* on a merry-go-round…



- For simplicity, let say you are *standing* still (remain stationary relative to the rotating frame) so that $\mathbf{v}'=0$ → there is no Coriolis force !

- Everything is perpendicular here. The fictitious centrifugal force is

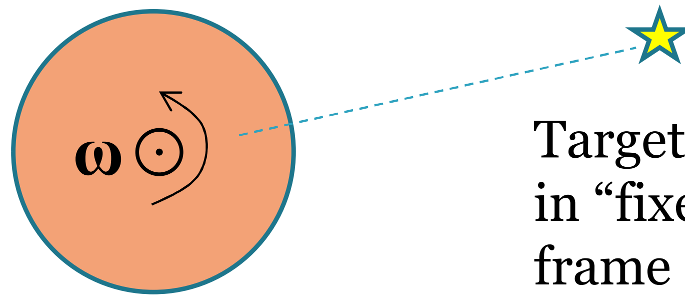
$$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') \quad\rightarrow\quad m\omega^2 r' = \frac{mv^2}{r'}(outward)$$

- **Note**: this fictitious centrifugal force ~ *r*' so that it gets larger as one moves out and vanishes at the origin.

Example of Fictitious Forces in a Rotating Frame

Now, we allow $\mathbf{v}' \neq 0$

Consider standing in the middle and throwing a rock directly at a fixed target fixed outside the merry-go-around

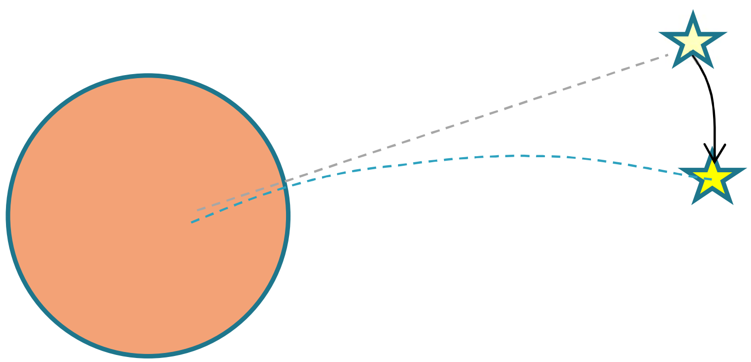


Target fixed in "fixed" frame

View in "fixed" frame

→ In the "fixed" inertial frame, the rock experiences no net force and it will go out toward the target along the *straight* path as shown.

- However, as the rocks moves outward, the thrower will rotate.

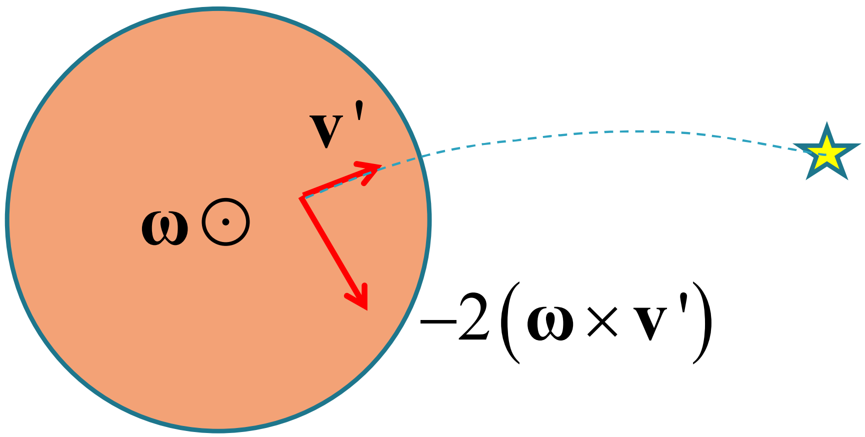


View in "rotating" frame

→ To the thrower in the "rotating" frame, the rock will look like veering off to right.

Example of Fictitious Forces in a Rotating Frame

So, within the “rotating” frame, there is a fictitious Coriolis force



View in “rotating” frame

Rule of Thumb: object deflects to the right of $\mathbf{v}'$

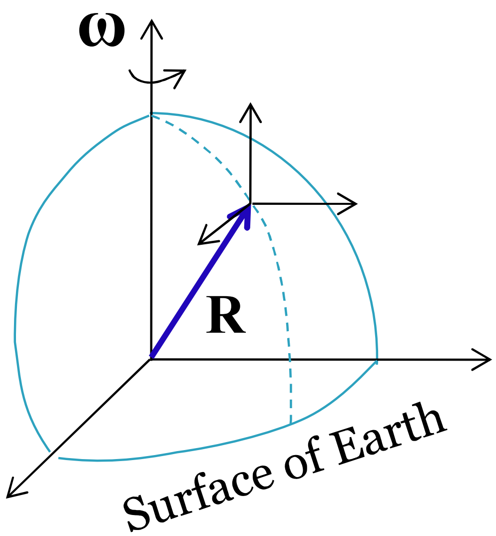
MOVIE: http://techtv.mit.edu/videos/3714-the-coriolis-effect
https://www.youtube.com/watch?v=dt_XJp77-mk

Earth as a Rotating Frame

The fact that we live on the surface of the Earth which rotates

→ Means that we are subject to these fictitious forces

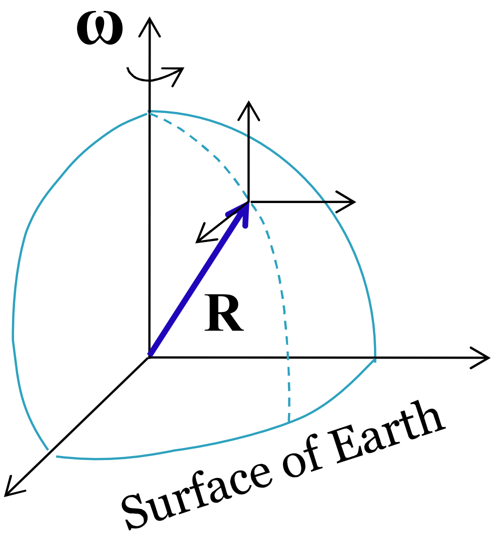
Additionally, for an object on the surface of the Earth, the origin of the "body" frame is not fixed so that **R** changes and we need to put back $\mathbf{a}_R$ but we will assume Earth to have a fixed rate of rotation, i.e., $\dot{\boldsymbol{\omega}} = 0$)



$$\mathbf{a}_f = \mathbf{a}_R + \mathbf{a}_r + 2(\boldsymbol{\omega}\times\mathbf{v}') + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')$$

Earth as a Rotating Frame

To calculate $\mathbf{a}_R$, consider $\mathbf{R}$...



→ It moves in the "fixed" frame but it is stationary in the "body" frame.

Considering the rate of change of **R** in the fixed frame, we have

$$\left(\frac{d\mathbf{R}}{dt}\right)_{fixed} = \cancel{\left(\frac{d\mathbf{R}}{dt}\right)_{body}} + \boldsymbol{\omega}\times\mathbf{R}$$

Taking the time derivative as measured in the fixed frame again, we have,

$$\ddot{\mathbf{R}} = \left(\frac{d\dot{\mathbf{R}}}{dt}\right)_{fixed} = \mathbf{a}_R = \cancel{\left(\frac{d\dot{\mathbf{R}}}{dt}\right)_{body}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$$

$$\Rightarrow \quad \mathbf{a}_R = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$$

Earth as a Rotating Frame

Plugging this result into our equation for $\mathbf{a}_f$, we have,

$$\mathbf{a}_f = \mathbf{a}_r + 2(\boldsymbol{\omega} \times \mathbf{v}') + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{r}'])$$

And the effective force in the rotating frame becomes,

$$\mathbf{F}_{eff} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}') - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{r}']) = m\mathbf{a}_r$$

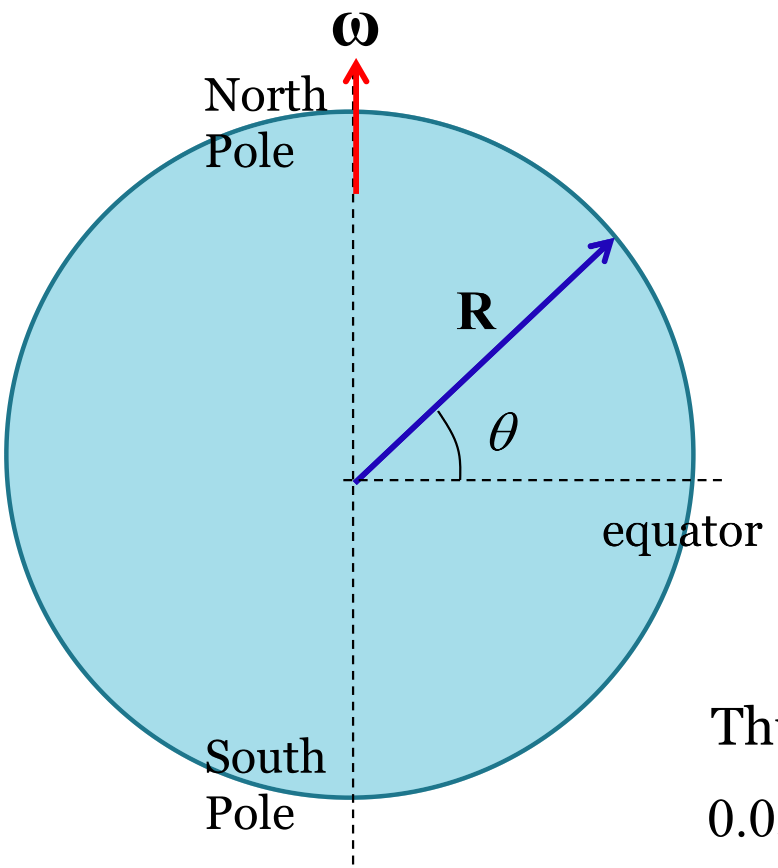
Letting $\mathbf{F} = m\mathbf{g}_0$ (gravity) be the only actual physical force, and assuming that we're interested in objects near the surface of the Earth so that $\mathbf{R} + \mathbf{r}' \simeq \mathbf{R}$

$$\mathbf{F}_{eff} = \left[m\mathbf{g}_0 - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \right] - 2m(\boldsymbol{\omega} \times \mathbf{v}')$$

This is the effective gravity **g** (actual gravity + centrifugal force)

Earth as a Rotating Frame

Now, let look closer at this effective gravity as a function of latitude θ on Earth,



- For $\theta = 90^\circ$ (North Pole),

$$\boldsymbol{\omega} \times \mathbf{R} = 0$$

→ Then, we have $\mathbf{g} = \mathbf{g}_0$

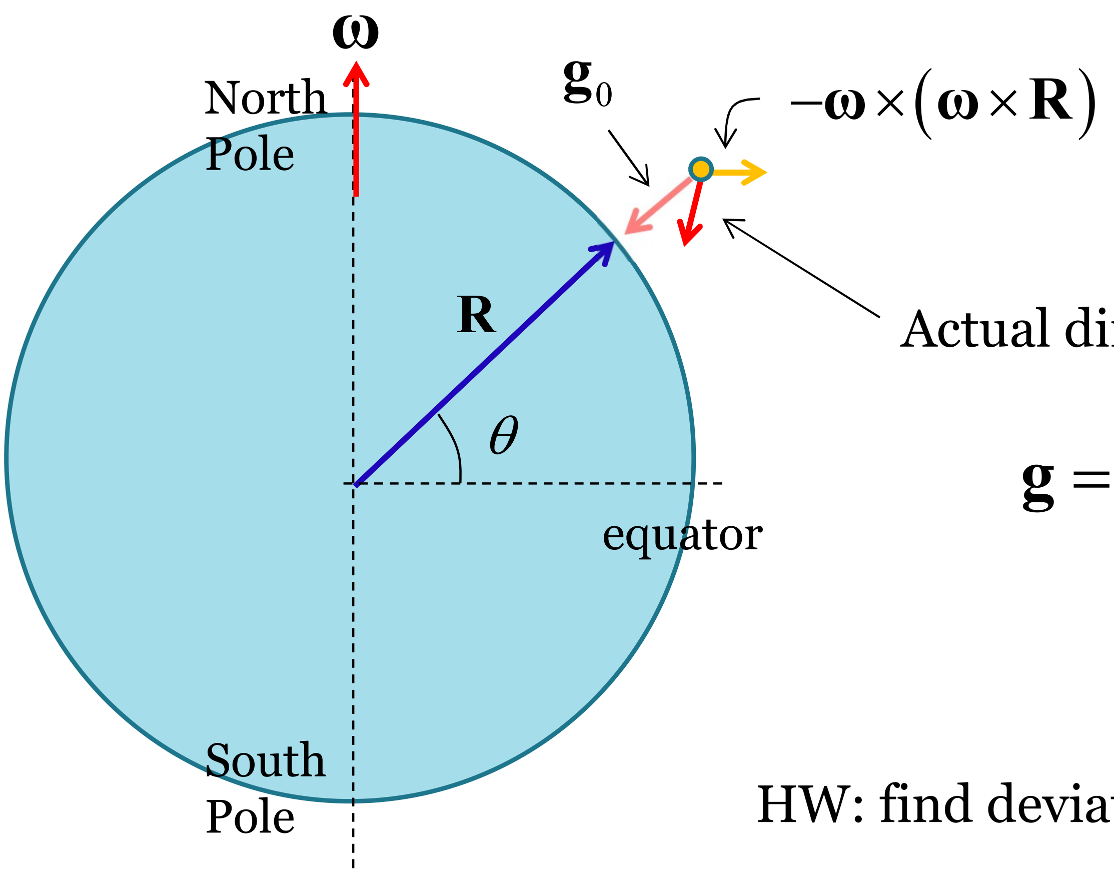
- For $\theta = 0^\circ$ (Equator),

$$\mathbf{g} = \mathbf{g}_0 + \omega^2 R\hat{\mathbf{R}} \qquad \left(\mathbf{g}_0 = -g\hat{\mathbf{R}}\right)$$

Thus, due to centrifugal effects, gravity g is about $0.052\, m/s^2$ less @ equator than @ poles

Earth as a Rotating Frame

In general, also notice that a mass on a string doesn't hang straight down:



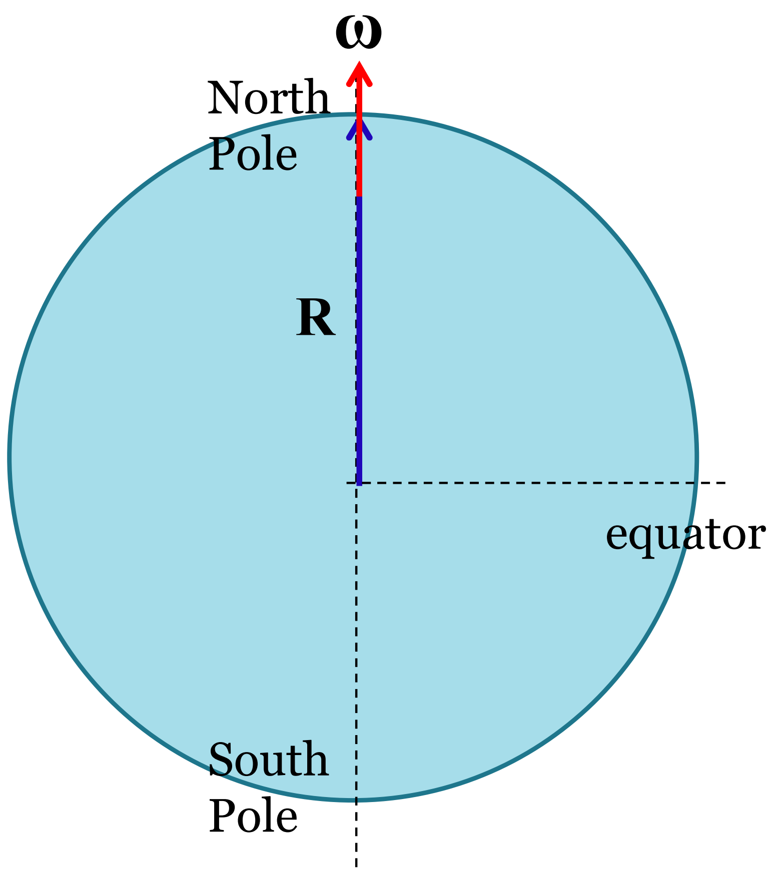
Actual direction of $\mathbf{g}$ (way exaggerated)

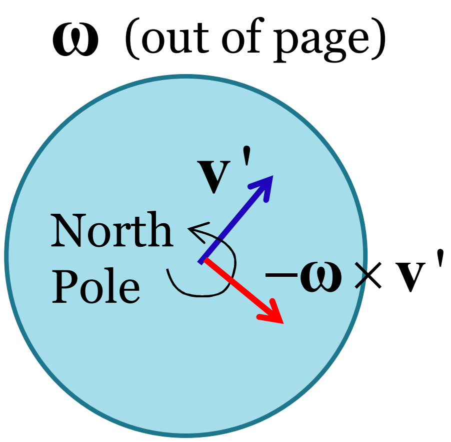
$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$$

HW: find deviation angle as a function of latitude θ

Coriolis Effect on Projectiles: $-2m(\boldsymbol{\omega}\times\mathbf{v}')$

Case 1: At North Pole



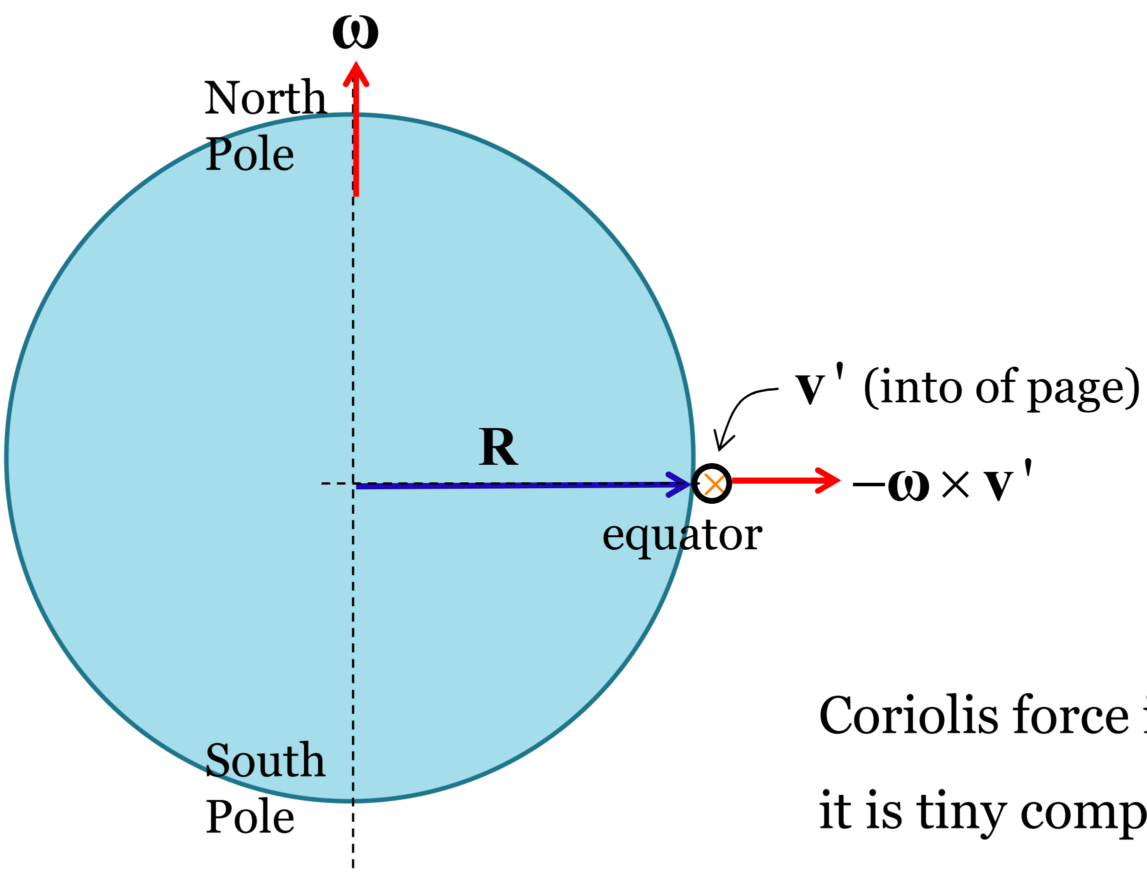


Top View

→ Projectile will defect to the right (as in the merry-go-round situation).

Coriolis Effect on Projectiles: $-2m(\boldsymbol{\omega}\times\mathbf{v}')$

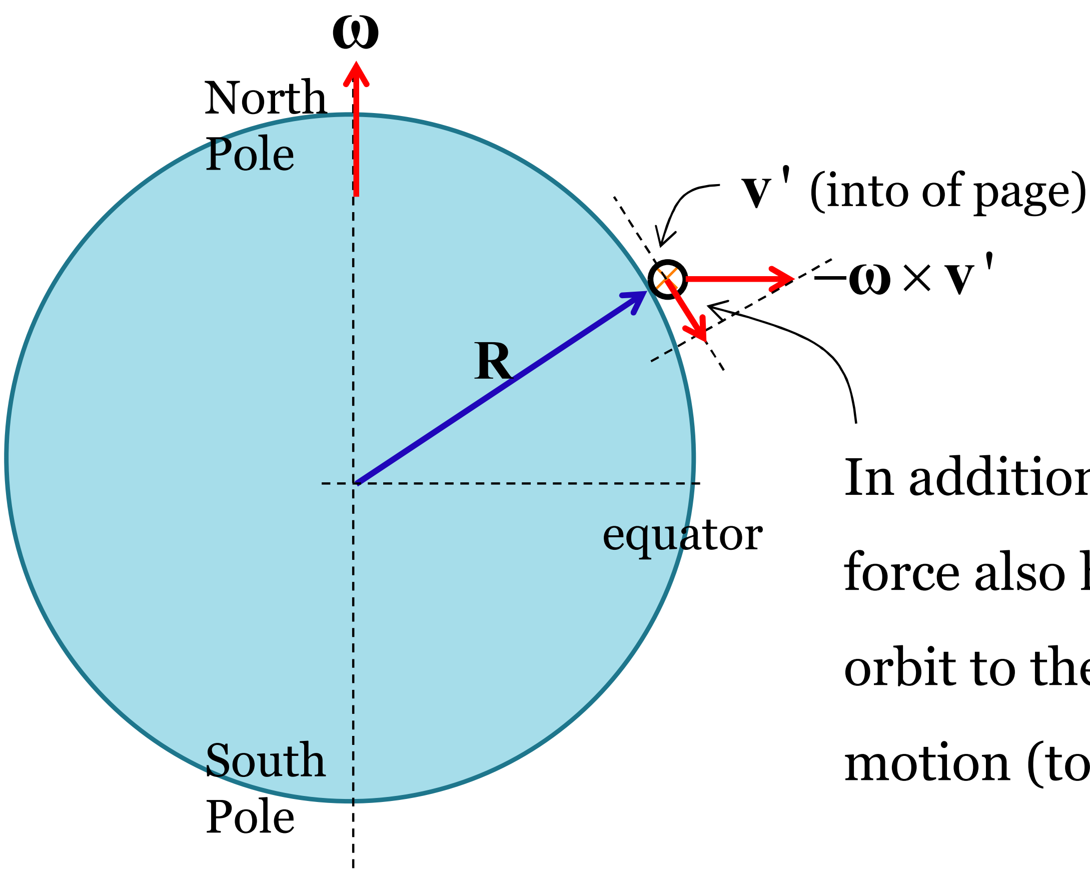
Case 2: At Equator → sending projectile to the East → $\mathbf{v}'$ (into of page)



Coriolis force is up (from the surface) – but it is tiny compared to regular gravity.

Coriolis Effect on Projectiles: $-2m(\boldsymbol{\omega}\times\mathbf{v}')$

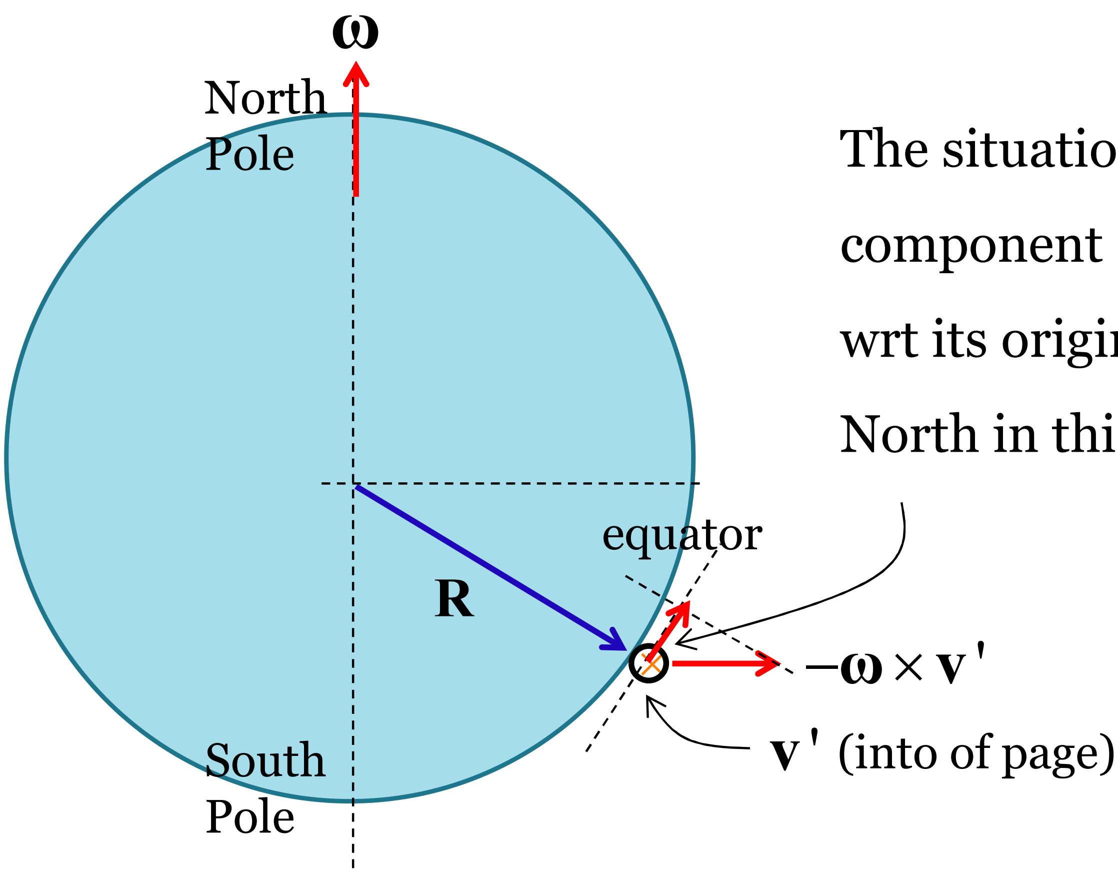
Case 3: At intermediate northern latitudes → sending projectile to the East



In addition to the up direction, Coriolis force also has a component deflecting the orbit to the right wrt its original direction of motion (toward South in this case).

Coriolis Effect on Projectiles: $-2m(\boldsymbol{\omega}\times\mathbf{v}')$

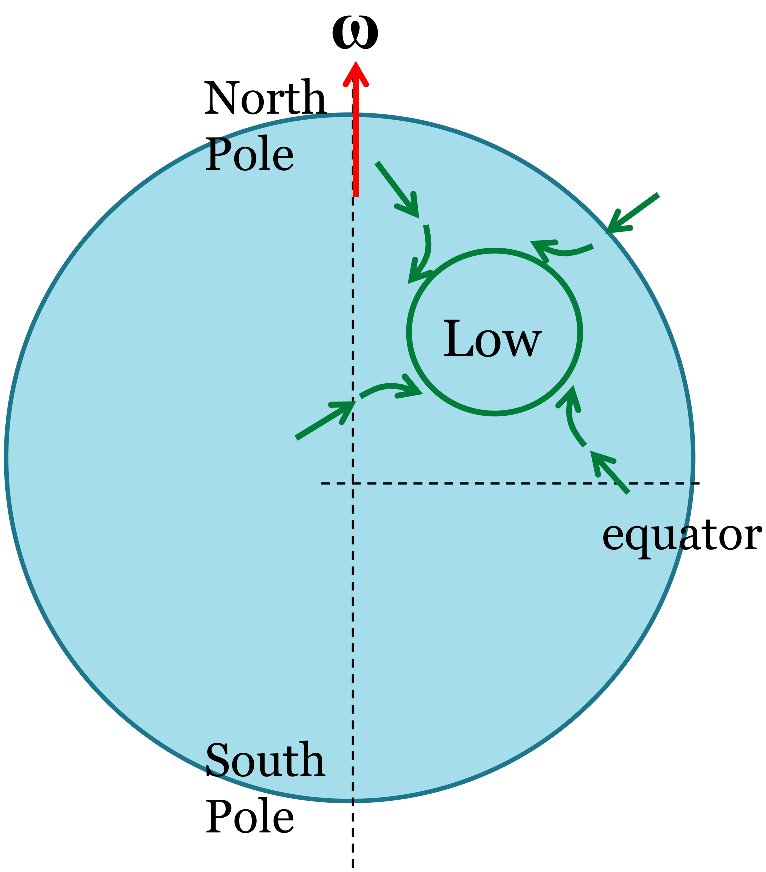
Case 4: Southern latitudes → sending projectile to the East



The situation is reverse. Coriolis force has a component deflecting the orbit to the left wrt its original direction of motion (toward North in this case).

Coriolis Effect on Projectiles: $-2m(\boldsymbol{\omega} \times \mathbf{v}')$

Coriolis effect on weather



Northern hemisphere – direction of wind will be defected to the right always resulting into forming a counter-clockwise circulating low pressure cell.

Northern hurricane (zeta)

Southern hurricane