# PHYS 705: Classical Mechanics

## Hamilton-Jacobi Equation

# Hamilton-Jacobi Equation

There is also a very elegant relation between the Hamiltonian Formulation of Mechanics and Quantum Mechanics.

To do that, we need to derive the Hamilton-Jacobi equation.

The Hamilton-Jacobi equation also represents a very general method in solving mechanical problems.

To start…

Let say we are able to find a canonical transformation taking our $2n$ phase space variables $(q_i, p_i)$ directly to $2n$ *constants of motion*, $(\beta_i, \alpha_i)$ i.e.,

$$Q_i = \beta_i \qquad P_i = \alpha_i$$

# Hamilton-Jacobi Equation

One sufficient condition to ensure that our new variables are constant in time is that the transformed Hamiltonian $K$ shall be identically zero.

If that is the case, the equations of motion will be,

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$$

Recall that we have the new and old Hamiltonian, $K$ and $H$, relating through the generating function $F$ by:

$$K = H + \frac{\partial F}{\partial t}$$

# Hamilton-Jacobi Equation

With $K$ identically equals to zero, we can write:

$$H\left(q,p,t\right)+\frac{\partial F}{\partial t}=0 \quad (*)$$

For convenience in later calculations, we shall take $F$ to be of Type 2 so that its independent variables are the old coordinates $q_i$ and the new momenta $P_i$. In the same notations as previously, we can write

$$F\left(q,Q,P,t\right)=F_2(q,P,t)-Q_iP_i$$

Now, in order to write $H$ in terms of this chosen set of variables $\left(q,P,t\right)$, we will replace all the $p_i$ in $H$ using the transformation equation:

$$p_i=\frac{\partial F_2\left(q,P,t\right)}{\partial q_i}$$

# Hamilton-Jacobi Equation

Then, one can formally rewrite Eq. (*) as:

$$H\left(q_1,\cdots,q_n;\frac{\partial F_2}{\partial q_1},\cdots,\frac{\partial F_2}{\partial q_n};t\right)+\frac{\partial F_2}{\partial t}=0$$

note that $\frac{\partial F}{\partial t}=\frac{\partial F_2}{\partial t}$ since

$F\left(q,P,t\right)=F_2(q,P,t)-Q_iP_i$

$$p_i=\frac{\partial F_2\left(q,\alpha,t\right)}{\partial q_i}$$

This is a PDE for $F_2$ and is known as the Hamilton-Jacobi Equation.

Notes: -Since $P_i=\alpha_i$ are constants, the HJ equation constitutes a partial differential equation of ($n$+1) independent variables: $\left(q_1,\cdots,q_n,t\right)$

- We used the fact that the new $P$'s and $Q$'s are constants but we have not specified in how to determine them yet.
- It is customary to denote the solution $F_2$ by $S$ and called it the Hamilton's Principal Function.

# Hamilton-Jacobi Equation

Suppose we are able to find a solution to this 1st order partial differential equation in ($n$+1) variables…

$$F_2 \equiv S = S\left(q_1, \cdots, q_n, \alpha_1, \cdots, \alpha_n; t\right)$$

where we have explicitly written out the constant new momenta: $P_i = \alpha_i$

Recall that for a Type 2 generating function, we have the following partial derivatives describing the canonical transformation:

$$\left(p_i = \frac{\partial S}{\partial q_i}\right) \qquad \Longrightarrow \qquad p_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial q_i} \quad (T1)$$

$$\left(Q_i = \frac{\partial S}{\partial P_i}\right) \qquad \qquad Q_i = \beta_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial \alpha_i} \quad (T2)$$

# Hamilton-Jacobi Equation

After explicitly taking the partial derivative on the RHS of Eq. (T1)

$$p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i}$$

and evaluating them at the *initial* time $t_0$, we will have *n* equations that one can invert to solve for the *n* unknown constants $\alpha_i$ in terms of the *initial conditions* $(q_{0,} p_0, t_0)$, i.e.,

$$\alpha_i = \alpha_i (q_{0,} p_0, t_0)$$

Similarly, by explicitly evaluating the partial derivatives on the RHS of Eq. (T2) at time $t_0$ , we obtain the other *n* constants of motion $\beta_i$

$$Q_i = \beta_i = \left. \frac{\partial S(q,\alpha,t)}{\partial \alpha_i} \right|_{q=q_0, t=t_0}$$

# Hamilton-Jacobi Equation

With all $2n$ constants of motion $\alpha_i, \beta_i$ solved, we can now use Eq. (T2) again to solve for $q_i$ in terms of the $\alpha_i, \beta_i$ at a later time *t*.

$$\left(\beta_i = \frac{\partial S(q,\alpha,t)}{\partial \alpha_i}\right) \Rightarrow \boxed{q_i = q_i(\alpha,\beta,t)}$$

Then, with $\alpha_i, \beta_i$, and $q_i$ known, we can use Eq. (T1) again to evaluate $p_i$ in terms of $\alpha_i, \beta_i$ at a later time *t*.

$$\left(p_i = \frac{\partial S(q(\alpha,\beta,t),\alpha,t)}{\partial q_i}\right) \Rightarrow \boxed{p_i = p_i(\alpha,\beta,t)}$$

➡ The two boxed equations constitute the desired complete solutions of the Hamilton equations of motion.

# Hamilton Principal Function and Action

Now, let consider the *total* time derivative of the Hamilton Principal function $S$,

$$\frac{dS(q,\alpha,t)}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}$$

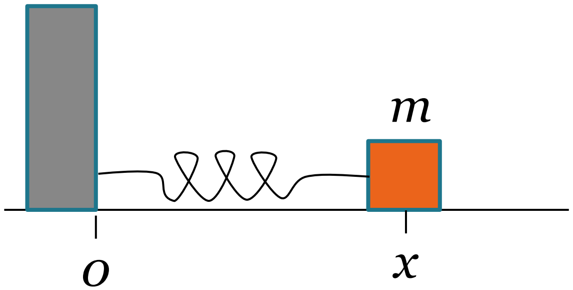
Using Eq. (T1) and the Hamilton-Jacobi equation itself, we can replace the two partial derivatives on the right by $p_i$ and $H$,

$$\frac{dS}{dt} = p_i\dot{q}_i - H = L \qquad \left(p_i = \frac{\partial S}{\partial q_i}\right) \quad \left(H + \frac{\partial S}{\partial t} = 0\right)$$

So, the Hamilton Principal Function is differed at most from the action by a constant, i.e.,

$$S = \int L\,dt + \text{constant}$$

# Example: Solving Harmonic Oscillator with HJ




$$f(q) = -kq \qquad U(q) = \frac{kq^2}{2}$$

Recall, we have:

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E \qquad \omega^2 = k/m$$

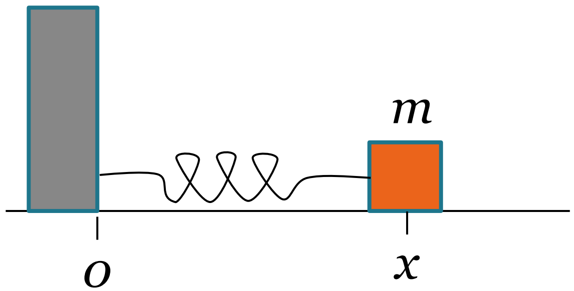
The Hamilton-Jacobi equation gives:

$$H\left(q, \frac{\partial S}{\partial q}; t\right) + \frac{\partial S}{\partial t} = 0 \quad \Longrightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

Recall that under this scheme, we have:

$$K = 0 \quad \Longrightarrow \quad Q = \beta \quad \text{and} \quad P = \alpha$$

# Example: Solving Harmonic Oscillator with HJ




$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E \qquad \omega^2 = k/m$$

$$f(q) = -kq \qquad U(q) = \frac{kq^2}{2}$$

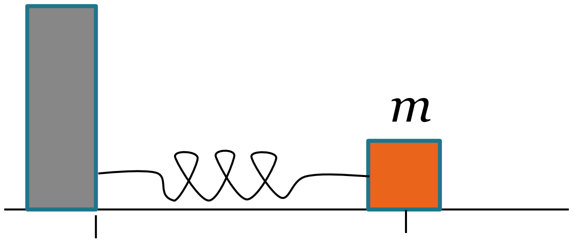
When $H$ does not explicitly depend on time, we can generally try this trial solution for $S$:

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t$$ ($q$ and $t$ are *separable*)

With this trial solution, we have $\partial S/\partial t = -\alpha$ and the HJ equation becomes:

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \Rightarrow \frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \qquad \left(\begin{array}{l} recall: LHS = H \\ \text{so } H = \alpha \end{array}\right)$$

# Example: Solving Harmonic Oscillator with HJ




The previous equation can be immediately integrated:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \quad \Rightarrow \quad \left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2 = 2m\alpha$$

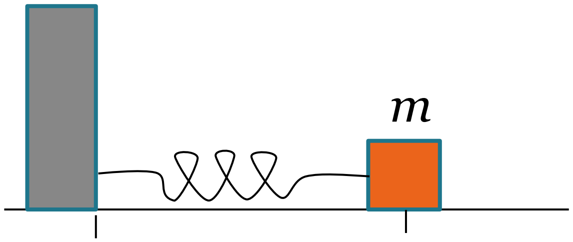
$$\left(\frac{\partial W}{\partial q}\right)^2 = 2m\alpha - m^2\omega^2 q^2$$

$$\Rightarrow \quad W = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq$$

and

$$S = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq - \alpha t$$

# Example: Solving Harmonic Oscillator with HJ




We don't need to integrate yet since we only need to use $\frac{\partial S}{\partial \alpha}, \frac{\partial S}{\partial q}$ in the transformation eqs:

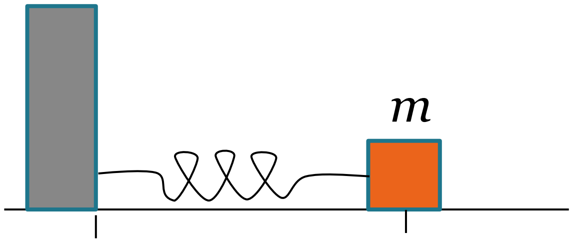
#1 eq: $\left(\beta_i = \frac{\partial S(q,\alpha,t)}{\partial \alpha_i}\right) \Longrightarrow Q = \beta = \frac{\partial S}{\partial \alpha} = \int \frac{\partial}{\partial \alpha}\left(\sqrt{2m\alpha - m^2\omega^2 q^2}\right) dq - t$

$$\beta = \int \left(\frac{1}{2}\right)(2m)\frac{dq}{\sqrt{2m\alpha - m^2\omega^2 q^2}} - t = \sqrt{\frac{m}{2\alpha}} \int \frac{dq}{\sqrt{1 - m\omega^2 q^2 / 2\alpha}} - t$$

Letting $x = \sqrt{\frac{m}{2\alpha}}\omega q$ , we can rewrite the integral as:

$$t + \beta = \frac{1}{\omega}\int \frac{dx}{\sqrt{1-x^2}} \Longrightarrow t + \beta = \frac{1}{\omega}\arcsin(x) = \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2\alpha}}\omega q\right)$$

# Example: Solving Harmonic Oscillator with HJ




$$t+\beta=\frac{1}{\omega}\arcsin(x)=\frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2\alpha}}\omega q\right)$$

Inverting the arcsin, we have

$$\boxed{q(t)=\sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t+\beta')}$$

with $\beta'=\omega\beta$

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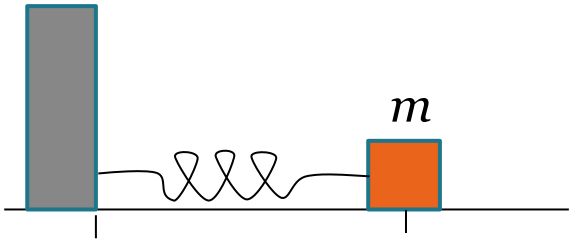
The 2nd transformation equation gives,

$$\left(p_i=\frac{\partial S(q(\alpha,\beta,t),\alpha,t)}{\partial q_i}\right)$$

#2 eq: ⇒

$$p=\frac{\partial S}{\partial q}=\frac{\partial}{\partial q}\left(\int\sqrt{2m\alpha-m^2\omega^2q^2}\,dq-\alpha t\right)=\sqrt{2m\alpha-m^2\omega^2q^2}$$

# Example: Solving Harmonic Oscillator with HJ




Substituting $q(t)$, we have,

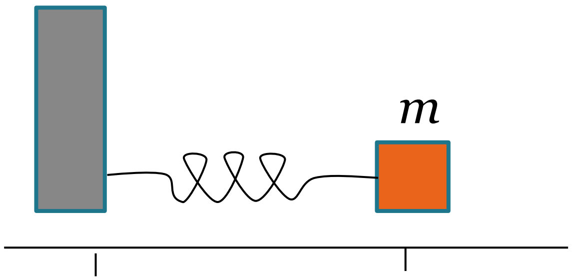
$$p(t) = \sqrt{2m\alpha\left(1 - \frac{m\omega^2}{2\alpha}\left(\frac{2\alpha}{m\omega^2}\right)\sin^2(\omega t + \beta')\right)}$$

$$\Rightarrow \quad p(t) = \sqrt{2m\alpha\left(1 - \sin^2(\omega t + \beta')\right)} \qquad \text{with } \beta' = \omega\beta$$

$$p(t) = \sqrt{2m\alpha\left(\cos^2(\omega t + \beta')\right)}$$

$$\boxed{p(t) = \sqrt{2m\alpha}\cos(\omega t + \beta')}$$

# Example: Solving Harmonic Oscillator with HJ




$$q(t)=\sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t+\beta')$$

$$p(t)=\sqrt{2m\alpha}\cos(\omega t+\beta')$$

Completing the calculations, we still need to link the two constants of motion $(\alpha,\beta)$ to initial conditions: @ $t=0,\quad q=q_0,\quad p=p_0$

Recall that the constant of motion $\alpha$ is $H$ (=$E$) of the system so that we can calculate this by squaring $q$ and $p$ at time $t=0$:

$$\alpha=\frac{1}{2m}\left(p_0^2+m^2\omega^2q_0^2\right)$$

And, we can calculate $\beta$ at time $t=0$ by dividing the 2 eqs: $\dfrac{q_0}{p_0}=\dfrac{1}{m\omega}\tan(\beta')$

# Hamilton Principal Function and Action

Most interestingly, let consider the *total* time derivative of the Hamilton Principal function *S*,

$$\frac{dS(q,\alpha,t)}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}$$

Using Eq. (T1) and the Hamilton-Jacobi equation itself, we can replace the two partial derivatives on the right by $p_i$ and $H$,

$$\frac{dS}{dt} = p_i\dot{q}_i - H = L \qquad \left(p_i = \frac{\partial S}{\partial q_i}\right) \quad \left(H + \frac{\partial S}{\partial t} = 0\right)$$

So, the Hamilton Principal Function is differed at most from the action by a constant, i.e.,

$$S = \int L\, dt + \text{constant}$$

# Another example in using the Hamilton-Jacobi Method (t-dependent *H*)

(G 10.8) Suppose we are given a Hamiltonian to a system as,

$$H = \frac{p^2}{2m} - mAtq \qquad \text{where } A \text{ is a constant}$$

Our task is to solve for the equation of motion by means of the Hamilton's principle function $S$ with the initial conditions

$$t = 0, \quad q_0 = 0, \quad p_0 = mv_0$$

The Hamilton-Jacobi equation for $S$ is:

$$H\left(q; \frac{\partial S}{\partial q}; t\right) + \frac{\partial S}{\partial t} = 0$$

# Example in using the Hamilton-Jacobi Method

Recall that the Hamilton's principle function $S$ is a Type 2 generating function with independent variables $(q, P, t)$

$$S = S(q, P, t)$$

with the condition that the canonically transformed variables are constants, i.e.,

$$Q = \beta \quad \text{and} \quad P = \alpha$$

For a Type 2 generating function, we have the following partial derivatives for $S$:

$$p = \frac{\partial S(q, \alpha, t)}{\partial q} \quad (T1) \qquad Q = \beta = \frac{\partial S(q, \alpha, t)}{\partial \alpha} \quad (T2)$$

# Example in using the Hamilton-Jacobi Method

Writing $H$ out explicitly in the Hamilton-Jacobi equation, we have

$$\underbrace{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - mAtq}_{H\left(q;\frac{\partial S}{\partial q};t\right)} + \frac{\partial S}{\partial t} = 0 \qquad (*)$$

Here, we assume a solution to be separatable of the form:

$$S(q,\alpha;t) = f(\alpha,t)q + g(t)$$

Substituting this into the HJ Eq (*) above, we have,

$$\frac{1}{2m} f(\alpha,t)^2 - mAtq + f'(\alpha,t)q + g'(t) = 0$$

( ' is partial time derivative here)

# Example in using the Hamilton-Jacobi Method

$$\frac{1}{2m} f(\alpha,t)^2 - mAtq + f'(\alpha,t)\, q + g'(t) = 0$$

Concentrating on the $q$ dependent terms, they have to be independently add up to zero… So, we have,

$$f'(\alpha,t) = mAt \quad \Rightarrow \quad f(\alpha,t) = \frac{mAt^2}{2} + f_0(\alpha)$$

And, requiring the remaining two terms adding up to zero also, we have

$$g'(t) = -\frac{1}{2m} f^2(t)$$

Substituting $f(t)$ from above, we have

$$g'(t) = -\frac{1}{2m}\left[\frac{mAt^2}{2} + f_0\right]^2 = -\frac{1}{2m}\left[\frac{m^2A^2t^4}{4} + mAt^2 f_0 + f_0^{\,2}\right]$$

# Example in using the Hamilton-Jacobi Method

Integrating wrt time on both sides, we then have,

$$g(t) = -\frac{mA^2t^5}{40} - \frac{Af_0}{6}t^3 - \frac{f_0^{\,2}}{2m}t + g_0$$

Putting both $f(t)$ and $g(t)$ back into the Hamilton's principle function, we have,

$$S(q,\alpha;t) = \left(\frac{mAt^2}{2} + f_0(\alpha)\right)q - \frac{mA^2t^5}{40} - \frac{Af_0(\alpha)}{6}t^3 - \frac{f_0^{\,2}(\alpha)}{2m}t + g_0$$

Since the Hamilton-Jacobi Equation only involves partial derivatives of $S$, $g_0$ can be taken to be zero without affect the dynamics and for simplicity, we will take the integration constant $f_0$ to be simply $\alpha$, i.e., $f_0(\alpha) = \alpha$

$$S(q,\alpha;t) = \left(\frac{mAt^2}{2} + \alpha\right)q - \frac{mA^2t^5}{40} - \frac{A\alpha}{6}t^3 - \frac{\alpha^2}{2m}t$$

# Example in using the Hamilton-Jacobi Method

We can apply Eq. T2 to solve for $\beta$,

$$S(q,\alpha;t) = \frac{mAt^2}{2}q + \alpha q - \frac{mA^2t^5}{40} - \frac{A\alpha}{6}t^3 - \frac{\alpha^2}{2m}t$$

$$Q = \beta = \frac{\partial S(q,\alpha,t)}{\partial \alpha} = q - \frac{A}{6}t^3 - \frac{\alpha}{m}t$$

Solving for $q$, we then have,

$$q(t) = \beta + \frac{A}{6}t^3 + \frac{\alpha}{m}t$$

From the initial conditions, we have $t = 0, q_0 = 0, p_0 = mv_0$

→ This implies that $q(0) = \beta = 0$

→ And, apply Eq. T1, we can solve for $\alpha$,

$$p(0) = \left.\frac{\partial S(q,\alpha,t)}{\partial q}\right|_{t=0} = \left[\frac{mAt^2}{2} + \alpha\right]_{t=0} = \alpha = mv_0$$

# Example in using the Hamilton-Jacobi Method

Putting these two constants $\alpha, \beta$ back into our equation for $q(t)$, we finally arrive at an explicit equation of motion for the system:

$$q(t) = \frac{A}{6}t^3 + v_o t$$

Using Eq. T1 again, we have ,

$$p(t) = \frac{\partial S(q,\alpha,t)}{\partial q} = \frac{mAt^2}{2} + \alpha$$

We just found that $\alpha = mv_0$ ,

$$p(t) = \frac{mAt^2}{2} + mv_0$$

# Connection to the Schrödinger equation in QM

(This connection was first derived by David Bohm.)

We first start by writing a quantum wavefunction $\psi(\mathbf{r},t)$ in phase-amplitude form,

$$\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$$

where $A(\mathbf{r},t)$ and $S(\mathbf{r},t)$ are the real amplitude and phase of $\psi(\mathbf{r},t)$

Let say this quantum particle is moving under the influence of a conservative potential $U$ and its time evolution is then given by the Schrodinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi$$

# Connection to the Schrödinger equation in QM

Now, we substitute the polar form of the wavefuntion into the Schrodinger equation term by term:

$$\psi(\mathbf{r},t) = A(\mathbf{r},t)\,e^{iS(\mathbf{r},t)/\hbar}$$

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$$i\hbar\frac{\partial\psi}{\partial t}: \qquad i\hbar\frac{\partial\psi}{\partial t} = e^{iS/\hbar}\left(i\hbar\frac{\partial A}{\partial t} - \cancel{\hbar}\frac{1}{\cancel{\hbar}}A\frac{\partial S}{\partial t}\right)$$

$$\nabla^2\psi: \qquad \nabla\psi = e^{iS/\hbar}\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

$$\nabla^2\psi = \left(\frac{i}{\hbar}e^{iS/\hbar}\nabla S\right)\cdot\left(\nabla A + \frac{i}{\hbar}A\nabla S\right) + e^{iS/\hbar}\nabla\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

$$= \frac{i}{\hbar}e^{iS/\hbar}\nabla S\cdot\nabla A - \frac{1}{\hbar^2}A(\nabla S)^2 + e^{iS/\hbar}\left(\nabla^2 A + \frac{i}{\hbar}\nabla A\cdot\nabla S + \frac{i}{\hbar}A\nabla^2 S\right)$$

$$= e^{iS/\hbar}\left(\nabla^2 A - \frac{A}{\hbar^2}(\nabla S)^2 + \frac{i}{\hbar}\left(2\nabla A\cdot\nabla S + A\nabla^2 S\right)\right)$$

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# Connection to the Schrödinger equation in QM

Substituting these terms into the Schrodinger equation, dividing out the common factor $e^{iS/\hbar}$ and separating them into real/imaginary parts,

IM (black): $i\hbar\partial\psi/\partial t$ $\quad\quad$ $\nabla^2\psi$

$$\hbar\frac{\partial A}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{1}{\hbar}\left(2\nabla A\cdot\nabla S + A\nabla^2 S\right)\right)$$

Simplifying and $\times 2A$:

$$2\frac{\partial A}{\partial t}A = -2\frac{1}{2m}\left(2A\nabla A\cdot\nabla S + AA\nabla^2 S\right)$$

Regrouping:

$$\frac{\partial A^2}{\partial t} = -\nabla\cdot\left(A^2\frac{\nabla S}{m}\right)$$

# Connection to the Schrödinger equation in QM

Now, for the real part of the equation, we have,

RE (blue):

$$\overbrace{-A\frac{\partial S}{\partial t}}^{i\hbar\partial\psi/\partial t}=-\frac{\hbar^2}{2m}\Bigg(\overbrace{\nabla^2 A-\frac{A}{\hbar^2}(\nabla S)^2}^{\nabla^2\psi}\Bigg)+\overbrace{UA}^{U\psi}$$

Simplifying: $$-\frac{\partial S}{\partial t}=-\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}+\frac{1}{2m}(\nabla S)^2+U$$

Rearranging: $$\boxed{\frac{1}{2m}(\nabla S)^2+U+\frac{\partial S}{\partial t}=\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}}$$

# Connection to the Schrödinger equation in QM

Taking the limit $\hbar \rightarrow 0$ , we have,

$$\frac{1}{2m}(\nabla S)^2 + U + \frac{\partial S}{\partial t} = 0$$

This is the Hamilton Jacobi equation if we identify the quantum phase $S$ of $\psi$ with the Hamilton Principal Function or the classical Action.

To see that explicitly, recall the Hamilton Jacobi equation is,

$$H\left( q_1, \cdots, q_n; \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_n}; t \right) + \frac{\partial S}{\partial t} = 0$$

# Connection to the Schrödinger equation in QM

Now, for a particle with mass $m$ moving under the influence of a conservative potential $U$, its Hamiltonian $H$ is given by its total energy:

$$H = \frac{p^2}{2m} + U$$

Using the "inverse" canonical transformation, Eq. (T1), we can write $H$ as,

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)\left(\frac{\partial S}{\partial q}\right) + U = \frac{1}{2m}\left(\nabla S\right)^2 + U \qquad \left(p = \frac{\partial S}{\partial q}\right)$$

Substituting this into the Hamilton-Jacobi Equation, we have,

$$\frac{1}{2m}\left(\nabla S\right)^2 + U + \frac{\partial S}{\partial t} = 0$$

(This is the same equation for the phase of the wavefunction from Schrödinger Eq)

# Connection to the Schrödinger equation in QM

Notes:

1. The neglected term proportional to $\hbar^2$ is called the Bohm's quantum potential,

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}$$

Note that this potential is *nonl*ocal due to its spatial diffusive term and it can be interpreted as the source of nonlocality in QM.

2. The imaginary part of the Schrödinger equation can be interpreted as the continuity equation for the conserved probability density $\rho = A^2 = \psi^* \psi$ with the velocity field given by $v = \nabla S / m = p / m$ :

$$\frac{\partial A^2}{\partial t} = -\nabla \cdot \left( A^2 \frac{\nabla S}{m} \right)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)$$

# Hamilton's Characteristic Function

Let consider the case when the Hamiltonian is constant in time but we don't a priori know what it is,

$$H\left(q_i, p_i\right) = \alpha_1$$

Now, let also consider a canonical transformation under which the new momenta are all constants of the motion,

$$P_i = \alpha_i$$

(the transformed $Q_i$ are not restricted a priori. )

AND we choose the new canonical momentum $\alpha_1$ to be the constant $H$ itself.

➡ Then, we seek to determine a *time-independent* generating function $W(q_i, P_i)$ (Type-2) producing the desired CT.

# Hamilton’s Characteristic Function

Similar to the development of the Hamilton’s Principal Function, since $W(q,P)$ is Type-2, the corresponding equations of transformation are

$$p_i = \frac{\partial W(q,\alpha)}{\partial q_i} \quad (T1)$$

$$Q_i = \frac{\partial W(q,\alpha)}{\partial \alpha_i} \quad (T2)$$

(Note: the indices inside $W(q, P)$ are being suppressed.)

Now, since $W(q,P)$ is time-independent, $\frac{\partial W(q,\alpha)}{\partial t} = 0$ and we have

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) + \cancel{\frac{\partial W}{\partial t}} = K = \alpha_1$$

# Hamilton's Characteristic Function

$W(q,P)$ is called the Hamilton's Characteristic Function and

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) - \alpha_1 = 0$$

is the partial differential equation (Hamilton-Jacobi Equation) for *W*.
Here, we have *n* independent constants $\alpha_i$ (with $\alpha_1 = H$) in determining this partial diff. eq.

And, through Eqs T1 and T2, *W* generates the desired canonical transformation in which all transformed momenta are constants !

# Hamilton's Characteristic Function

In the transformed variables $(Q_i, P_i)$, $K = \alpha_1$, and the EOM is given by the Hamilton's Equations,

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \qquad \text{or} \qquad P_i = \alpha_i \qquad \text{(as required)}$$

$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

By integrating, this immediately gives,

$$Q_1 = t + \beta_1$$

$$Q_i = \beta_i, \quad i \neq 1$$

where $\beta_i$ are some integration constants determined by ICs.

Note that $Q_1$ is basically time and its conjugate momenta $P_1 = \alpha_1 = K$ is the Hamiltonian.

# Hamilton's Characteristic Function

By solving the H-J Eq, we can obtain $W(q,\alpha)$

Then, to get a solution for $(q_i, p_i)$, we use the transformation equations (T1 & T2),

$$p_i = \frac{\partial W(q_i,\alpha_i)}{\partial q_i} \qquad Q_i = \frac{\partial W(q_i,\alpha_i)}{\partial \alpha_i} = \begin{cases} t+\beta_1, & i=1 \\ \beta_i, & i \neq 1 \end{cases}$$

$p_i$ can be directly evaluated by taking the derivatives of $W$

$q_i$ can be solved by taking the derivatives of $W$ on the RHR and inverting the equation

The set of $2n$ constants $(\alpha_i, \beta_i)$ are fixed through the $2n$ initial conditions $\{q_i(0), p_i(0)\}$

# Hamilton's Characteristic Function

When the Hamiltonian does not dependent on time explicitly, one can use either

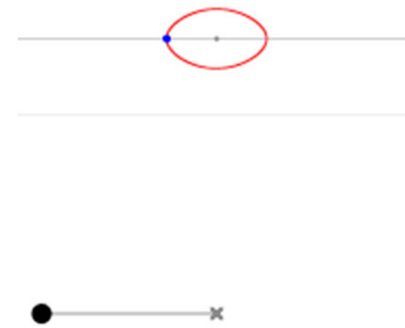
- The Hamilton Principal Function $S(q,P,t)$ or
- The Hamilton Characteristic Function $W(q,P)$

to solve a particular mechanics problem using the H-J equation and they are related by:

$$S(q,P,t) = W(q,P) - \alpha_1 t$$

**Note**: As we have seen earlier, the Hamilton Principal Function can be used when *H* dep on time explicitly but not the HCF.

# Action-Angle Variables in 1dof



- Often time, for a system which oscillates in time, we might not be interested in the details about the EOM but we just want information about the *frequencies* of its oscillations.

- The H-J procedure in terms of the Hamilton Characteristic Function can be a powerful method in doing that.

- To get a sense on the power of the technique, we will examine the simple case when we have only one degree of freedom.

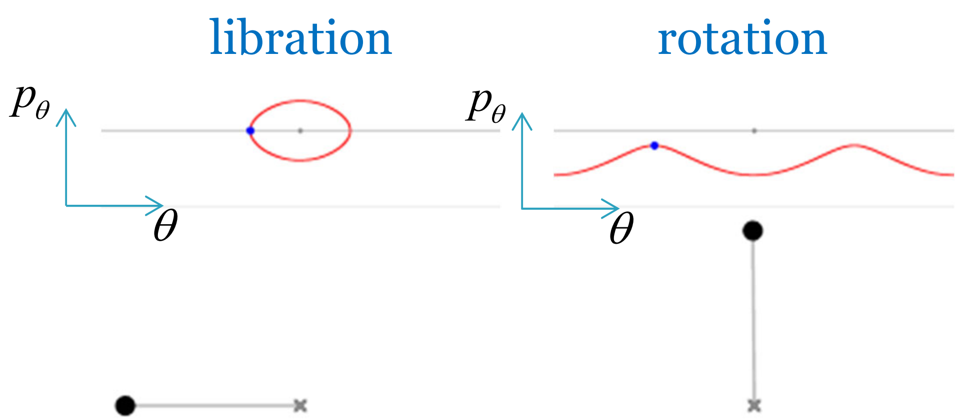
- We continue to assume a conservative system with $H = \alpha_1$ being a constant

# Action-Angle Variables in 1dof

Let say we know that the dynamic of a system is *periodic* so that

$$q(t+T)=q(t)$$

Recall from our discussion for a pendulum on dynamical systems, we have two possible periodic states:



# Action-Angle Variables in 1dof

- Now, we introduce a new variable

$$J = \oint p \, dq$$

called the **Action Variable**, where the path integral is taken over one full cycle of the periodic motion.

- Now, since the Hamiltonian is a constant, we have

$$H(q,p) = \alpha_1$$

- Then, by inverting the above equation to solve for *p*, we have

$$p = p(q, \alpha_1)$$

# Action-Angle Variables in 1dof

- Then, we "integrate out" the $q$ dependence in

$$J = \oint p(q, \alpha_1)\, dq$$

One can then write that $J$ is a function of $\alpha_1$ alone or vice versa,

$$\alpha_1 \equiv H = H(J)$$

Since $J$ is a function of the constant $\alpha_1$ alone, it is itself a constant.

- Now, instead of requiring our new momenta $P$ to be $\alpha_1$ , we requires

$P = J$ (another constant instead of $\alpha_1$ )

# Action-Angle Variables in 1dof

- Then, our Hamilton Characteristic Function can be written as

$$W = W(q, J)$$

- From the transformation equations, the generalized coordinate $Q$ corresponding to $P$ is given by Eq. T2

$$Q = \frac{\partial W}{\partial P}$$

- Enforcing our new momenta $P$ to be $J$ and calling its conjugate coordinate $w$, we have

$$w = \frac{\partial W}{\partial J}$$

- Here, in this context, $w$ is called the **Angle Variable**.

# Action-Angle Variables in 1dof

- Note, since the generating function $W(q, P)$ is time independent, the Hamiltonian in the transformed coordinate $K = H$ so that

$$\alpha_1 \equiv H = H(J) = K(J)$$

- Correspondingly, using the Hamilton Equations, the EOM for $w$ is,

$$\dot{w} = \frac{\partial K(J)}{\partial J} = v(J)$$

- Now, since $K$ is a constant, its partial derivative with $J$ will also be a constant function of $J$, calling it $v(J)$

$$\frac{\partial K(J)}{\partial J} \equiv v(J)$$

# Action-Angle Variables in 1dof

$$\dot{w} = \nu(J)$$

The above diff. eq. can be immediately integrated to get

$$w = \nu t + \beta$$

where $\beta$ is an integration constant depending on IC.

- Thus, this **Angle Variable** $w$ is a linear function of time !

# Action-Angle Variables in 1dof

Now, let integrate $w$ over one period $T$ of the periodic motion.

$$\Delta w = \oint_T dw = \oint_T \frac{\partial w}{\partial q} dq$$

Using the transformation equation, $w = \dfrac{\partial W}{\partial J}$, we then have,

$$\Delta w = \oint_T \frac{\partial^2 W}{\partial q \partial J} dq$$

# Action-Angle Variables in 1dof

Since $J$ is a constant with no $q$ dependence, we can move the derivative wrt $J$ outside of the $q$ integral,

$$\Delta w = \oint_T \frac{\partial^2 W}{\partial J \partial q}\, dq = \frac{d}{dJ} \oint_T \frac{\partial W}{\partial q}\, dq$$

Then, using the other transformation equation, $p = \dfrac{\partial W}{\partial q}$, we can write,

$$\Delta w = \frac{d}{dJ} \oint_T p\, dq = \frac{dJ}{dJ} = 1$$

(recall $J = \oint p\, dq$ )

# Action-Angle Variables in 1dof

$$\Delta w = 1$$

This result means that $w$ changes by *unity* as $q$ goes through a complete period $T$. Then, since we have $w = \nu t + \beta$, we can write

$$\Delta w = w(T) - w(0) = \nu T = 1$$

- $\nu(J) = 1/T$ gives the frequency of the periodic oscillation associated with $q$ !
- and it can be directly evaluated thru

$$\nu(J) = \frac{\partial K(J)}{\partial J} \qquad (*)$$

without finding the complete EOM

# Action-Angle Variables in 1dof: Example

Let consider a linear harmonic oscillator given by the Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2 = \alpha$$

Motion is periodic in $q$

$\alpha$ is the const total $E$

$\omega = \sqrt{k/m}$ is the natural freq

Solving for $p$, we have,

$$p = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

Then, we can substitute this into the integral for the Action Variable,

$$J = \oint_T \sqrt{2m\alpha - m^2\omega^2 q^2}\; dq$$

# Action-Angle Variables in 1dof: Example

Using the following coordinate change,

$$q=\sqrt{\frac{2\alpha}{m\omega^2}}\sin\theta$$

The integrand can be simplified,

$$\sqrt{2m\alpha-m^2\omega^2q^2}\,dq=\sqrt{2m\alpha-m^2\omega^2\left(\frac{2\alpha}{m\omega^2}\right)\sin^2\theta}\sqrt{\frac{2\alpha}{m\omega^2}}(\cos\theta d\theta)$$

$$=\sqrt{2m\alpha\left(1-\sin^2\theta\right)}\sqrt{\frac{2\alpha}{m\omega^2}}(\cos\theta d\theta)$$

$$=\sqrt{2m\alpha\cos^2\theta}\sqrt{\frac{2\alpha}{m\omega^2}}(\cos\theta d\theta)=\frac{2\alpha}{\omega}\cos^2\theta d\theta$$

# Action-Angle Variables in 1dof: Example

Putting this back into our integral for the Action Variable, we have,

$$J = \frac{2\alpha}{\omega}\int_0^{2\pi} \cos^2\theta d\theta$$

This gives,

$$J = \frac{2\alpha}{\omega}\pi$$

Inverting to solve for $\alpha$ in terms of $J$, we have,

$$\alpha \equiv H = K = \frac{J\omega}{2\pi}$$

# Action-Angle Variables in 1dof: Example

Finally, applying our result for $\nu$ using Eq. (*), we have

$$\nu = \frac{\partial K}{\partial J} = \frac{\partial}{\partial J}\left(\frac{J\omega}{2\pi}\right) = \frac{\omega}{2\pi}$$

Thus,

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Here, we have derived the frequency of the linear harmonic oscillator by only calculating the Action Variable without explicitly solving for the EOM !