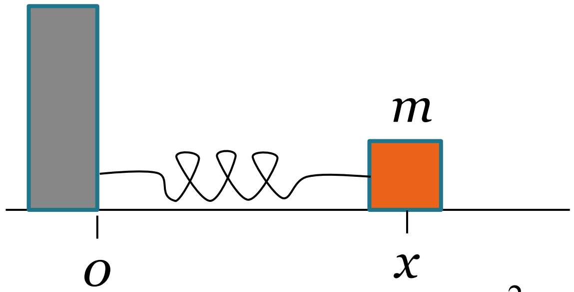
PHYS 705: Classical Mechanics

Canonical Transformation II

Example: Harmonic Oscillator



$$f(x) = -kx \quad U(x) = \frac{kx^2}{2}$$

$$L = T - U = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \to \quad \dot{x} = \frac{p}{m}$$

$$H = p\dot{x} - L = p\dot{x} - \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}$$

$$= \frac{p^2}{m} - \frac{m}{2}\frac{p^2}{m^2} + \frac{kx^2}{2} \quad (\text{in } x \;\&\; p)$$

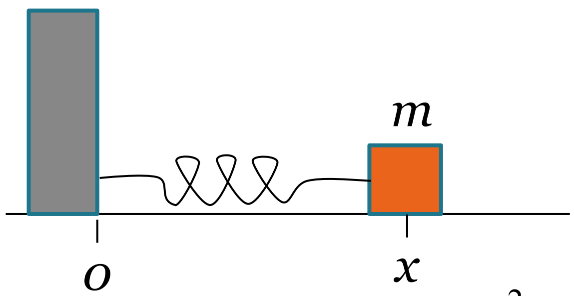
$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Define $\omega = \sqrt{k/m}$ or $\omega^2 m = k$ →

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$$

Since U does not dep on $\dot{x}$ and $x \mapsto x$ does not dep t explicitly, $H = E$.

Example: Harmonic Oscillator



$$f(x) = -kx \quad U(x) = \frac{kx^2}{2}$$

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$$

Application of the Hamilton's Equations give,

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{2m^2\omega^2}{2m}x = -m\omega^2 x$$

Combining the two equations, we have the standard equation for SHO: $\ddot{x} + \omega^2 x = 0$

Notice that the system is not cyclic with the original variable x. We will attempt to find a canonical transformation from (x, p) to (X, P) such that X is cyclic so that P is constant.

Example: Harmonic Oscillator

With the form of H as sums of squares $H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$, we will try to exploit $\sin^2 + \cos^2 = 1$

→ Let try $p = g(P)\cos(X) \qquad x = \frac{g(P)}{m\omega}\sin(X)$ (*)

Substituting these into our Hamiltonian, we have

X is cyclic in K!

$$K = \frac{1}{2m}\left(g^2(P)\cos^2 X + m^2\omega^2 \frac{g^2(P)}{m^2\omega^2}\sin^2 X\right) = \frac{g^2(P)}{2m}$$

Now, we need to find the right $g(P)$ such that the transformation is canonical.

→ Try the following type 1 generating function: $F = F(x, X, t)$

(with x and X being the indep vars)

Example: Harmonic Oscillator

As in previous examples, we will try to write p (now the dep var for F) in terms of the (indep var) (x, X) by dividing the two equations in CT:

$$\frac{p}{x} = \frac{\cancel{g(P)} \cos X}{\cancel{g(P)} \sin X \left(1/m\omega\right)} = m\omega \cot X \qquad \text{or} \qquad p = m\omega x \cot X$$

For F to be a canonical transformation, it has to satisfy the partial derivative relations (Type 1 in Table 9.1 in G):

$$p = \frac{\partial F}{\partial x} \qquad\qquad P = -\frac{\partial F}{\partial X}$$

The first partial derivative equation (left) gives:

$$\frac{\partial F}{\partial x} = p = m\omega x \cot X \quad \Longrightarrow \quad F = F(x, X, t) = \frac{m\omega x^2}{2} \cot X$$

Example: Harmonic Oscillator

$$F = F(x, X, t) = \frac{m\omega x^2}{2}\cot X$$

With this trial solution for F, we can substitute it into the 2nd partial derivative relation to get an expression for the other CT:

$$P = -\frac{\partial F}{\partial X} \quad \Rightarrow \quad P = -\frac{m\omega x^2}{2}\left(-\frac{1}{\sin^2 X}\right) = \frac{m\omega x^2}{2}\frac{1}{\sin^2 X}$$

Solving for x (old) in term of (X, P) (new), we have:

$$x^2 = 2\frac{P}{m\omega}\sin^2 X \quad \Rightarrow \quad x = \sqrt{\frac{2P}{m\omega}}\sin X$$

Example: Harmonic Oscillator

To get the second transformation for p (old) in terms of $\left(X,P\right)$ (new), we can use our previous relation $\left(p = m\omega x \cot X\right)$ and our newly derived $x\left(X,P\right)$ eq:

$$x = \sqrt{\frac{2P}{m\omega}} \sin X$$

$$p = m\omega\, x \cot X$$

$$p = m\omega\left(\sqrt{\frac{2P}{m\omega}} \sin X\right)\cot X = \sqrt{2m\omega P}\cos X$$

$$\boxed{p = \sqrt{2m\omega P}\cos X}$$

Example: Harmonic Oscillator

Since we explicitly calculated this pair of transformations $\{x(X,P),\, p(X,P)\}$ using the appropriate generating function, they are canonical:

$$x = \sqrt{\frac{2P}{m\omega}}\sin X \qquad p = \sqrt{2m\omega P}\cos X$$

By comparing with our original transformation equation (Eq. (*)), we have:

$$\begin{cases} x = \dfrac{g(P)}{m\omega}\sin X \\ p = g(P)\cos X \end{cases} \quad \text{gives} \quad g(P) = \sqrt{2m\omega P} \qquad (*')$$

Then, the transformed Hamiltonian is: $K = \dfrac{g^2(P)}{2m} = \dfrac{2m\omega P}{2m} = \omega P$

Example: Harmonic Oscillator

$$\frac{\partial F}{\partial t} = 0$$

With the transformed Hamiltonian $K = \omega P = H = E$, the Hamilton's Equations give,

$$\dot{P} = -\frac{\partial K}{\partial X} = 0 \quad (X \text{ is cyclic})$$

$$\rightarrow \boxed{P = const}$$

$$\dot{X} = \frac{\partial K}{\partial P} = \omega$$

$$\rightarrow \boxed{X = \omega t + \alpha}$$

depends on IC

**Notice how simple the EOM are in the new transformed cyclic variables. In most application, the goal is to find a new set of canonical variables so that there are as many cyclic variables as possible.

**P is just the rescaled total energy E and X is just a rescaled and shifted time.

**(P, X) is an example of an Action-Angle pair of canonical variables.

Example: Harmonic Oscillator

Using the inverse transform:

$$x = \sqrt{\frac{2P}{m\omega}} \sin X \qquad\qquad p = \sqrt{2m\omega P} \cos X$$

The EOM in (X, P) can be written in terms of the original variable:

$$X = \omega t + \alpha \qquad\qquad P = const$$

$$x = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \alpha)$$

$$p = \sqrt{2m\omega P} \cos(\omega t + \alpha)$$

Example: Harmonic Oscillator

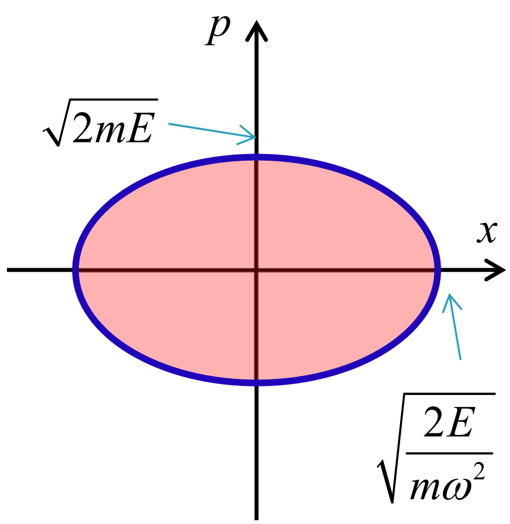
Then with $K = \omega P = E$

$$x = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

$$p = \sqrt{2mE} \cos(\omega t + \alpha)$$

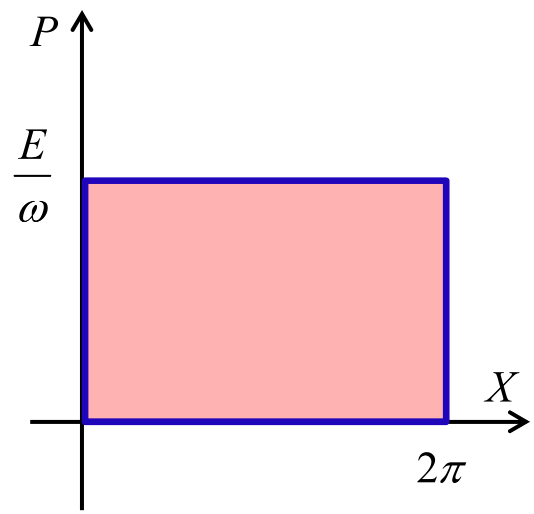
$$X = \omega t + \alpha$$

$$P = const$$



Phase Space area is invariant!

$$area = \frac{2\pi E}{\omega}$$



"Symplectic" Approach & Poisson Bracket

Let consider a time-independent canonical transformation:

$$Q_j = Q_j(q, p)$$
$$P_j = P_j(q, p)$$

(We assume the transformation not explicitly depending on time)

Now, let consider the full-time derivative of Q_j :

$$\dot{Q}_j = \frac{dQ_j}{dt} = \frac{\partial Q_j}{\partial q_i}\dot{q}_i + \frac{\partial Q_j}{\partial p_i}\dot{p}_i$$

(E's sum rule over *i*)

Using the Hamilton's equations, we can rewrite the equation as:

$$\dot{Q}_j = \frac{\partial Q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial H}{\partial q_i} \quad (*)$$

(E's sum rule over *i*)

"Symplectic" Approach & Poisson Bracket

Now, if the transformation is canonical, we also must have

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} = \frac{\partial H}{\partial P_j}$$

(Note: $K = H + \cancel{\partial F/\partial t} = H$ since canonical trans does not dep on time explicitly)

Using the inverse transform $q_j = q_j(Q,P),\ p_j = p_j(Q,P)$, we can consider *H* (the original Hamiltonian) as a function of *Q* and *P*, i.e,

$$H = H\left(q\left(Q,P\right), p\left(Q,P\right)\right)$$

Then, we can evaluate the RHS of the equation for $\dot{Q}_j$ in the 1st line as:

$$\dot{Q}_j = \frac{\partial H}{\partial P_j} = \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial P_j} + \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial P_j} \quad (**)$$

"Symplectic" Approach & Poisson Bracket

Comparing the two expressions for $\dot{Q}_j$, Eq. (*) and (**),

$$\dot{Q}_j = \frac{\partial Q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial H}{\partial q_i} \quad (*) \qquad \dot{Q}_j = \frac{\partial q_i}{\partial P_j}\frac{\partial H}{\partial q_i} + \frac{\partial p_i}{\partial P_j}\frac{\partial H}{\partial p_i} \quad (**)$$

we must have the following equalities for all (i, j) pairs:

$$\left(\frac{\partial Q_j}{\partial q_i}\right)_{q,p} = \left(\frac{\partial p_i}{\partial P_j}\right)_{Q,P} \quad \text{AND} \quad \left(\frac{\partial Q_j}{\partial p_i}\right)_{q,p} = -\left(\frac{\partial q_i}{\partial P_j}\right)_{Q,P}$$

(subscripts denote the functional dependence of the quantities)

Similar considerations for $\dot{P}_j$ will give a different but similar pairs of relations:

$$\left(\frac{\partial P_j}{\partial q_i}\right)_{q,p} = -\left(\frac{\partial p_i}{\partial Q_j}\right)_{Q,P} \quad \text{AND} \quad \left(\frac{\partial P_j}{\partial p_i}\right)_{q,p} = \left(\frac{\partial q_i}{\partial Q_j}\right)_{Q,P}$$

"Symplectic" Approach & Poisson Bracket

$$\left(\frac{\partial Q_j}{\partial q_i}\right)_{q,p} = \left(\frac{\partial p_i}{\partial P_j}\right)_{Q,P} \qquad \left(\frac{\partial Q_j}{\partial p_i}\right)_{q,p} = -\left(\frac{\partial q_i}{\partial P_j}\right)_{Q,P}$$

$$\left(\frac{\partial P_j}{\partial q_i}\right)_{q,p} = -\left(\frac{\partial p_i}{\partial Q_j}\right)_{Q,P} \qquad \left(\frac{\partial P_j}{\partial p_i}\right)_{q,p} = \left(\frac{\partial q_i}{\partial Q_j}\right)_{Q,P}$$

These four conditions for various (i, j) are called the "direct conditions" for a canonical transformation.

Poisson Bracket

For any two function $u(x,y)$ and $v(x,y)$ depending on x and y the Poisson Bracket is defined as:

$$[u,v]_{x,y} \equiv \left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial v}{\partial y_j}\right)-\left(\frac{\partial u}{\partial y_j}\right)\left(\frac{\partial v}{\partial x_j}\right)$$

(E's sum rule for n dof)

PB is analogous to the Commutator in QM:

$$\frac{1}{i\hbar}[\![u,v]\!] \equiv \frac{1}{i\hbar}(uv-vu)$$

where u and v are two QM operators

Some Basic Property of Poisson Brackets

General properties for Poisson brackets with respect to an arbitrary pair of variables (x, y) (not necessarily canonical variables):

- $[u,u]=0$ (the x, y subscripts are suppressed here)
- $[u,v]=-[v,u]$ (anti-symmetric)
- $[au+bv,w]=a[u,w]+b[v,w]$ (linearity)
- $[uv,w]=[u,w]v+u[v,w]$ (distributive)
- $[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$ (Jacobi's Identity)

u, v, w are some arbitrary functions of (x, y) and a & b are some constants.

"Symplectic" Approach & Poisson Bracket

Now, if we substitute the canonical variables themselves as (u, v) & (x, y) , we have:

- $\left[q_j, q_k\right]_{q,p} = \left[p_j, p_k\right]_{q,p} = 0$
- $\left[q_j, p_k\right]_{q,p} = -\left[p_j, q_k\right]_{q,p} = \delta_{jk}$

(hw)

These are called the **Fundamental Poisson Brackets.**

And, under a canonical transformation $(q, p) \rightarrow (Q, P)$, we can also show that:

- $\left[Q_j, Q_k\right]_{q,p} = \left[P_j, P_k\right]_{q,p} = 0$
- $\left[Q_j, P_k\right]_{q,p} = -\left[P_j, Q_k\right]_{q,p} = \delta_{jk}$

The **Fundamental Poisson Brackets** are invariant under a CT !

Fundamental Poisson Brackets & CT

(E's sum rule on i index)

$$\left[Q_j,Q_k\right]_{q,p}=\frac{\partial Q_j}{\partial q_i}\frac{\partial Q_k}{\partial p_i}-\frac{\partial Q_j}{\partial p_i}\frac{\partial Q_k}{\partial q_i}=-\frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial P_k}-\frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial P_k}=-\frac{\partial Q_j}{\partial P_k}=0$$

$$\left[P_j,P_k\right]_{q,p}=\frac{\partial P_j}{\partial q_i}\frac{\partial P_k}{\partial p_i}-\frac{\partial P_j}{\partial p_i}\frac{\partial P_k}{\partial q_i}=\frac{\partial P_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k}+\frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k}=\frac{\partial P_j}{\partial Q_k}=0$$

$$\left[Q_j,P_k\right]_{q,p}=\frac{\partial Q_j}{\partial q_i}\frac{\partial P_k}{\partial p_i}-\frac{\partial Q_j}{\partial p_i}\frac{\partial P_k}{\partial q_i}=\frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k}+\frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k}=\frac{\partial Q_j}{\partial Q_k}=\delta_{jk}$$

$$\left[P_j,Q_k\right]_{q,p}=-\left[Q_k,P_j\right]_{q,p}=-\delta_{jk}$$

Using the Direct Conditions of the CT

The **Fundamental Poisson Brackets** are invariant under a CT !

"Symplectic" Approach & Poisson Bracket

A general Poisson Bracket $[u,v]_{q,p}$ is invariant under a CT: $(q,p)\rightarrow(Q,P)$ (time-invariant case) !

$$[u,v]_{Q,P} \equiv \frac{\partial u}{\partial Q_i}\frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i}\frac{\partial v}{\partial Q_i}$$

$$= \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial Q_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial P_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial P_i}\right) - \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial P_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial Q_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial Q_i}\right)$$

$\vdots$ multiply out and regroup

$$= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial q_k}[q_j,q_k]_{Q,P} + \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}[q_j,p_k]_{Q,P} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}[q_k,p_j]_{Q,P} + \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial p_k}[p_j,p_k]_{Q,P}$$

$$= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}\delta_{jk} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}\delta_{jk} = \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} = [u,v]_{q,p}$$

“Symplectic” Approach & Poisson Bracket

The invariant of the Poisson brackets under the CT of the canonical variables defines the “symplectic” structure of phase space.

$$[u,v]_{Q,P} = [u,v]_{q,p}$$

$$\left[Q_j, P_k\right]_{q,p} = \left[q_j, p_k\right]_{Q,P} = \delta_{jk}$$

Thus, in general, if we are calculating a Poisson Bracket with respect to a pair of canonical variable, we can drop the q,p subscript.

$$[u,v]_{q,p} \Rightarrow \quad [u,v]$$

"Symplectic" Approach & Poisson Bracket

Note that this symplectic structure for the canonical transformation can also be expressed elegantly using the matrix notation that we have introduced earlier :

Recall that the Hamilton Equations can be written in a matrix form,

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}$$

with $\eta_j = q_j, \qquad \eta_{j+n} = p_j; \qquad j = 1,\cdots,n$

$$\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_j = \frac{\partial H}{\partial q_j}, \qquad \left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_{j+n} = \frac{\partial H}{\partial p_j}; \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

"Symplectic" Approach & Poisson Bracket

Now, let $\boldsymbol{\zeta}$ the 2*n*-column vector corresponding to the *new* set of canonical coordinates Q_j, P_j

And, we can write a transformation from (*q*, *p*) → (*Q*, *P*) as a vector equation:

$\boldsymbol{\eta}$ $\boldsymbol{\zeta}$

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$$

Then, by chain rule, the time derivatives of the variables are related by:

$$\dot{\boldsymbol{\zeta}} = \mathbf{M}\dot{\boldsymbol{\eta}} \qquad \text{where} \qquad M_{jk}\dot{\eta}_k = \frac{\partial \zeta_j}{\partial \eta_k}\dot{\eta}_k, \quad j,k = 1,\cdots,2n$$

(M_{jk} is the Jacobian matrix)

Substituting the Hamilton Equation back into above, we have:

$$\dot{\boldsymbol{\zeta}} = \mathbf{MJ}\frac{\partial H}{\partial \boldsymbol{\eta}}$$

"Symplectic" Approach & Poisson Bracket

Considering the inverse transformation (Q, P) → (q, p) and by chain rule, we can rewrite the right most factor as: $\boldsymbol{\zeta}$ $\boldsymbol{\eta}$

$$\frac{\partial H}{\partial \eta_j} = \frac{\partial H}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial \eta_j}$$ sum over k

Previously,

$$\frac{\partial \zeta_j}{\partial \eta_k}\dot{\eta}_k = M_{jk}\dot{\eta}_k \qquad M_{jk} \; \dot{\eta}_k \quad \begin{bmatrix} j,k \rightarrow & \\ \downarrow & \end{bmatrix}\begin{bmatrix} k \\ \downarrow \end{bmatrix}$$

Here,

$$\frac{\partial H}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial \eta_j} = \frac{\partial H}{\partial \zeta_k}M_{kj} \qquad \partial H/\partial \zeta_k \; M_{kj} \quad \begin{bmatrix} k \rightarrow \end{bmatrix}\begin{bmatrix} k,j \rightarrow & \\ \downarrow & \end{bmatrix}$$

So, in matrix notation:

$$\frac{\partial H}{\partial \boldsymbol{\eta}} = \mathbf{M}^T \frac{\partial H}{\partial \boldsymbol{\zeta}}$$

Putting this back into our dynamic equation, we have,

$$\dot{\boldsymbol{\zeta}} = \mathbf{MJM}^T \frac{\partial H}{\partial \boldsymbol{\zeta}}$$

"Symplectic" Approach & Poisson Bracket

Now, if the coordinate transformation $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ is canonical, then the Hamiltonian equation must also be satisfied in the new set of coordinates:

$$\dot{\boldsymbol{\zeta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\zeta}}$$

Comparing this equation with our previous equation: $\dot{\boldsymbol{\zeta}} = \mathbf{MJM}^T \frac{\partial H}{\partial \boldsymbol{\zeta}}$

We can see that in order for $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ to be a valid canonical transformation, we need to have **M** satisfying the following condition:

$$\boxed{\mathbf{MJM}^T = \mathbf{J}}$$

(this condition is typically easier to use than the direction condition for a CT)

"Symplectic" Approach & Poisson Bracket

In terms of these matrix notation, we can also write the Poisson bracket as,

$$[u, v]_{\boldsymbol{\eta}} = \frac{\partial u}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial v}{\partial \boldsymbol{\eta}}$$

And if $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ is a canonical transformation, then Fundamental Poisson Brackets can simply written as,

$$[\boldsymbol{\zeta}, \boldsymbol{\zeta}]_{\boldsymbol{\eta}} = [\boldsymbol{\eta}, \boldsymbol{\eta}]_{\boldsymbol{\zeta}} = \mathbf{J}$$

Poisson Bracket & Dynamics

Consider some physical quantity u as a function of the PS variables $u = u(q, p, t)$:

- Its full time derivative is given by:

$$\frac{du}{dt} = \frac{\partial u}{\partial q_j}\dot{q}_j + \frac{\partial u}{\partial p_j}\dot{p}_j + \frac{\partial u}{\partial t}$$

- Applying the Hamilton's Equations: $\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$

$$\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial H}{\partial q_j}}_{[u,H]_{q,p}} + \frac{\partial u}{\partial t}$$

Poisson Bracket & Dynamics

So, we arrived at the following general equation of time evolution for $u(t)$:

$$\dot{u} = \frac{du}{dt} = \left[u, H\right] + \frac{\partial u}{\partial t}$$

→ We have explicitly written this equation without reference to any particular set of canonical variables to emphasize the fact that the form of the equation is *invariant* to any canonical transformations

→ Obviously, we need to use the appropriate Hamiltonian in the given coordinates.

Applying the above equation with $u = H$, we have $\left[H, H\right] = 0$ and:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Poisson Bracket & Dynamics

General Comments:

1. If $u = q_j \ or \ p_j$, we get back the Hamilton's Equations:

$$\dot{q}_j = \left[q_j, H \right] = \frac{\partial H}{\partial p_j} \qquad \text{and} \qquad \dot{p}_j = \left[p_j, H \right] = -\frac{\partial H}{\partial q_j} \qquad \text{(hw)}$$

2. If u is a constant of motion, i.e., $\dfrac{du}{dt} = 0$

$$\dot{u} = \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \quad \Longrightarrow \quad [u, H] = -\frac{\partial u}{\partial t}$$

Conversely, for u explicitly not depends on time, i.e., $\dfrac{\partial u}{\partial t} = 0$,

$$\frac{du}{dt} = 0 \quad \Longleftrightarrow \quad [u, H] = 0$$

Poisson Bracket & Dynamics

The PB is analogous to the Commutator in QM:

CM		QM	
$[u,H]$	⟺	$\frac{1}{i\hbar}[\![u,H]\!]$	(u is some quantum observable here)
(Poisson Bracket)		(QM Commutator)	

→ The Heisenberg equation of motion for u in QM is given by:

$$i\hbar\frac{du}{dt}=[\![u,H]\!]$$

(assume u has no explicit time dependence, i.e., $\frac{\partial u}{\partial t}=0$)

→ The statement on the vanishing of the Poisson bracket for u being a constant of motion in CM (a conserved quantity) corresponds to the statement that u commute with H in QM : $[\![u,H]\!]=0$

Poisson Bracket & Dynamics

3. Poisson's Theorem

- Let u, v be two constants of motion, i.e., $[u,H]=[v,H]=0$

Then, $[u,v]$ is another constant of motion ! (Jacob's Identity)

Check: $$[[u,v],H]=-[H,[u,v]]=[u,[v,H]]+[v,[H,u]]$$

$$=[u,\cancel{[v,H]}]-[v,\cancel{[u,H]}]=0$$

4. Assume that u does not explicitly depend on time so that

$$\frac{du}{dt}=[u,H] \quad (*)$$

- Now, let Taylor's expand $u(t)$ around $u(0)$ for t small:

$$u(t)=u(0)+t\left.\frac{du}{dt}\right|_0+\frac{t^2}{2!}\left.\frac{d^2u}{dt^2}\right|_0+\cdots$$

Poisson Bracket & Dynamics

- Using Eq (*), we have $\left.\frac{du}{dt}\right|_0 = [u, H]_0$ (evaluated at t = 0)

And, $\left.\frac{d^2u}{dt^2}\right|_0 = \left.\frac{d}{dt}\left(\frac{du}{dt}\right)\right|_0 = \left[[u, H], H\right]_0$ (evaluated at t = 0)

- Substituting these two terms and similar ones into our Taylor's expansion for $u(t)$, we have,

$$u(t) = u(0) + t[u, H]_0 + \frac{t^2}{2!}\left[[u, H], H\right]_0 + \cdots$$

Poisson Bracket & Dynamics

$$u(t) = u(0) + t\left[u, H\right]_0 + \frac{t^2}{2!}\left[\left[u, H\right], H\right]_0 + \cdots$$

➡ So, one can formally write down the time evolution of $u(t)$ as a series solution in terms of the Poisson brackets evaluated at $t = 0$!

The Hamiltonian is the **generator** of the system's motion in time !

➡ This has a direct correspondence to the QM interpretation of H:

→ The above Taylor's expansion can be written as an "operator" eq:

$$u(t) = e^{\hat{H}t} u(0)$$

where $\hat{H}\, u(0) \equiv \left[u, H\right]\Big|_0$

⟺

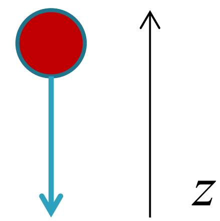
$$\left|u(t)\right\rangle = e^{iHt/\hbar} \left|u(0)\right\rangle$$

↑ (QM propagator)

Poisson Bracket & Dynamics

A simple example in applying to a freely falling particle: $u = z$

$$H = \frac{p^2}{2m} + mgz$$



$$z(t) = z(0) + t\left[z, H\right]_0 + \frac{t^2}{2!}\left[\left[z, H\right], H\right]_0 + \cdots$$

Now, we evaluate the different PBs:

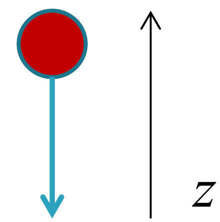
$$\left[z, H\right] = \frac{\partial z}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial z}{\partial p}\frac{\partial H}{\partial z} = \frac{p}{m}$$

$$\left[\left[\left[z, H\right], H\right], H\right] = 0$$

$$\left[\left[z, H\right], z\right] = \frac{\partial\left[z, H\right]}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial\left[z, H\right]}{\partial p}\frac{\partial H}{\partial z} = -\frac{1}{m}\cdot mg = -g$$

Poisson Bracket & Dynamics

$$H = \frac{p^2}{2m} + mgz$$



At $t = 0$, we have initial conditions: $z(0) = z_0, p(0) = p_0$

$$[z,H]_0 = \frac{p_0}{m} \qquad [[z,H],H]_0 = -g \qquad [[[z,H],H],H]_0 = 0$$

Substituting them into the expansion,

$$z(t) = z(0) + t[z,H]_0 + \frac{t^2}{2!}[[z,H],H]_0 + \cdots$$

We then have,

$$z(t) = z_0 + \frac{p_0}{m}t - \frac{g}{2}t^2$$