PHYS 705: Classical Mechanics

Central Force Problems II

Orbits in Central Force Problem

Suppose we're interested more in the **shape** of the orbit,
(not necessary the time evolution)

Then, a solution for $r = r(\theta)$ or $\theta = \theta(r)$ would be more useful!

First, let try to get $r = r(\theta)$:

Start with the r EOM: $$m\ddot{r} = \frac{l^2}{mr^3} - \frac{dV}{dr}$$

(**NOTE**: switch to V notation since we will be using u as inverse radius later)

We also have the angular momentum equation:

$$l = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}$$

$\times dt$ both sides → $$l\,dt = mr^2 d\theta$$

→ $$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

Orbits in Central Force Problem

Substituting this relation into our *r* equation, we have,

$$m\frac{d}{dt}\left(\frac{dr}{dt}\right)-\frac{l^2}{mr^3}=-\frac{dV}{dr}$$

$$\cancel{m}\frac{l}{\cancel{m}r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right)-\frac{l^2}{mr^3}=-\frac{dV}{dr} \qquad \left(\frac{d}{dt}=\frac{l}{mr^2}\frac{d}{d\theta}\right)$$

$$\frac{l^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{l^2}{mr^3}=-\frac{dV}{dr}$$

To simplify this further, it would be useful to introduce a coord trans: $u=\frac{1}{r}$

RHS: by chain rule, we have $-\frac{dV(r)}{dr}=-\frac{du}{dr}\frac{dV}{du}$

but, $\frac{du}{dr}=-\frac{1}{r^2}$ So → $=+\frac{1}{r^2}\frac{dV}{du}=u^2\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]$

Orbits in Central Force Problem

$$\frac{l^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{l^2}{mr^3}$$

Now, the **LHS**: $\frac{l^2}{m}u^2\frac{d}{d\theta}\left(u^2\frac{d}{d\theta}\left(\frac{1}{u}\right)\right)-\frac{l^2}{m}u^3$

$$=\frac{l^2}{m}u^2\left[\frac{d}{d\theta}\left(\cancel{u^2}\left(-\frac{1}{\cancel{u^2}}\right)\frac{du}{d\theta}\right)-u\right]$$

$$=\frac{l^2}{m}u^2\left[-\frac{d^2u}{d\theta^2}-u\right]$$

Putting the two sides together, we have

$$-\frac{l^2}{m}\cancel{u^2}\left[\frac{d^2u}{d\theta^2}+u\right]=\cancel{u^2}\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]$$

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2}\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]$$

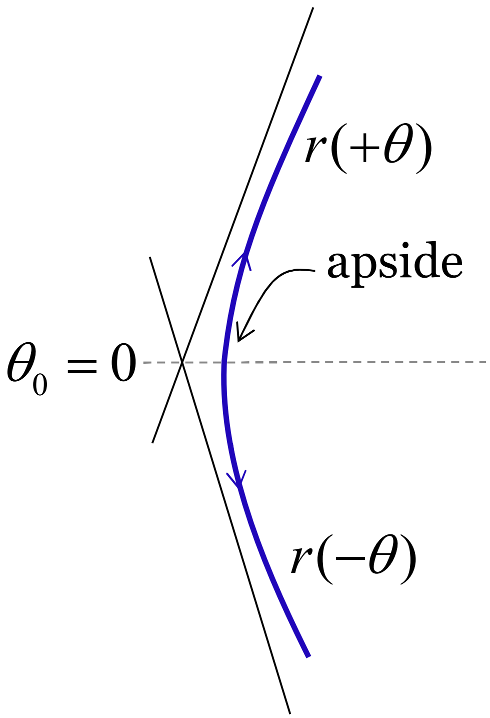
ODE for $u(\theta)$ or $r(\theta)$

Orbits in Central Force Problem

In this form, we can get a qualitative insight into the orbit's symmetry:

Consider a turning point (apside) at $\theta_0 = 0$ with ICs: $u(0) = u_0$ and $\left(du/d\theta\right)_0 = 0$

(turning pt)



Integrate the previous equation forward let say in the $+\theta$ direction and we get $r(+\theta)$

Now, since with $\theta' = -\theta$, we still have $\frac{d^2}{d\theta'^2} = \frac{d^2}{d\theta^2}$

Integrate the *same* ODE backward in the -θ direction with the same ICs will give the same *r* values, i.e.,

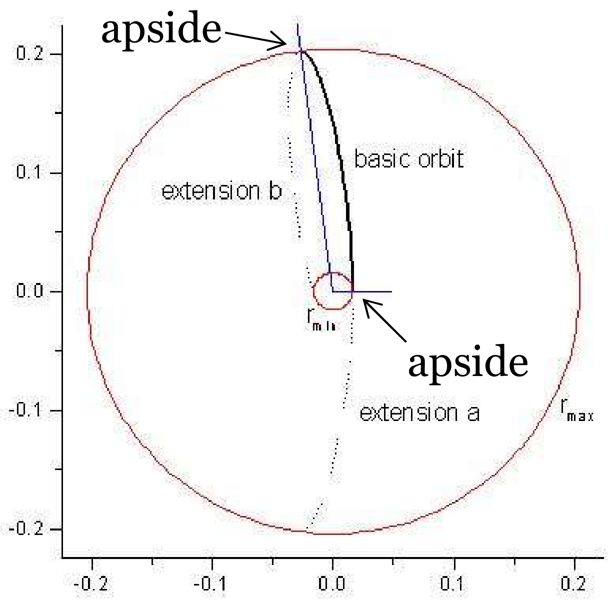
$$r(+\theta) = r(-\theta)$$

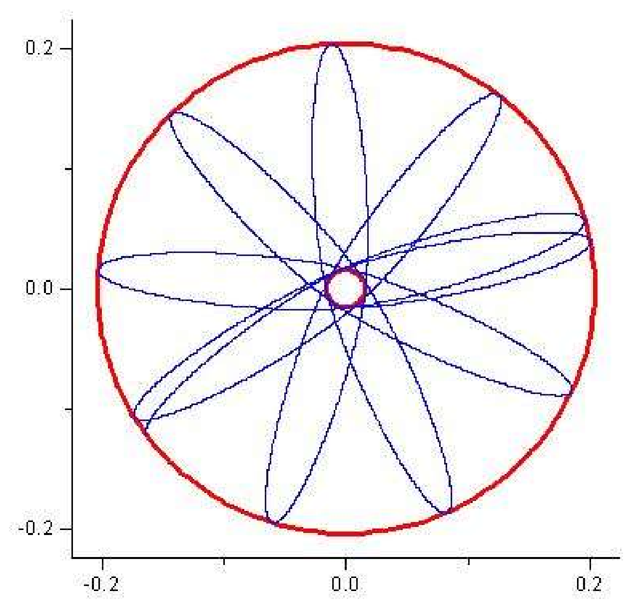
Orbit is symmetric about the apsides!

Orbits in Central Force Problem

So, we only need to find the orbit from one apside to the NEXT.

Then, to construct the full orbit, one can reflect this basic segment along the axis connecting the apside and the origin symmetrically.





Orbits in Central Force Problem

If you express the ODE back in terms of r and the force $F(r)$, you get,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{m}{l^2}\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]=-\frac{m}{l^2}r^2F(r)$$

recall:

$$F(r)=-\frac{dV(r)}{dr}=-\frac{du}{dr}\frac{dV}{du}$$

$$=\left(\frac{1}{r^2}\right)\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]$$

This form is useful if you want to solve for the force law for a given known orbit $r=r(\theta)$. (homework)

Example: let $r=ke^{\alpha\theta}$ ($k>0$ for physical orbits, r needs to be +)

(the orbit is a spiral: $+\alpha$ out and $-\alpha$ in)

Plug in 1st term in ODE:

$$\frac{d}{d\theta}\left(\frac{1}{ke^{\alpha\theta}}\right)=\frac{1}{k}\frac{d}{d\theta}\left(e^{-\alpha\theta}\right)=-\frac{\alpha}{k}e^{-\alpha\theta}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{ke^{\alpha\theta}}\right)=\frac{\alpha^2}{k}e^{-\alpha\theta}=\frac{\alpha^2}{r}$$

Orbits in Central Force Problem

So, we have

$$\frac{\alpha^2}{r}+\frac{1}{r}=-\frac{m}{l^2}r^2F(r)$$

$$-\left(\frac{\alpha^2+1}{r^3}\right)\left(\frac{l^2}{m}\right)=F(r)$$

So, for the prescribed orbit $r=ke^{\alpha\theta}$, the required force law is:

$$F(r)=-\frac{l^2\left(\alpha^2+1\right)}{mr^3}$$

(an inverse cubic force law)

Orbits in Central Force Problem

$$\frac{l^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{l^2}{mr^3}=-\frac{dV}{dr}$$

Instead of solving for $r=r(\theta)$ from the previous 2nd order ODE

There is also an alternative way to get the inverse orbit equation $\theta=\theta(r)$ by solving a quadrature.

Recall the *r* equation obtained from conservation of energy equation:

$$\dot{r}=\frac{dr}{dt}=\sqrt{\frac{2}{m}\left(E-V(r)-\frac{l^2}{2mr^2}\right)}$$

To eliminate *t* in the equation, note that $\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}$ by chain rule again,

This can be rewritten using the angular momentum equation,

$$\frac{dr}{dt}=\frac{d\theta}{dt}\frac{dr}{d\theta}=\frac{l}{mr^2}\frac{dr}{d\theta}\quad(note: mr^2\dot{\theta}=l)$$

Orbits in Central Force Problem

Substituting this into our $\dot{r}$ equation, we have

$$\frac{l}{mr^2}\frac{dr}{d\theta} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

Rearranging terms and integrating both sides gives,

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

The right hand side can be integrated by quadrature.

Orbits in Central Force Problem

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

Comments:

1. If $F(r) \sim r^n, i.e., V(r) \sim r^{n+1}$

   Then, the integral can be integrated (in closed form) in several cases:

   n =1, -2, -3: can be solved in terms of trig functions.

   n = -2 → is the Kepler's problem
   n = 1 → is the harmonic oscillator

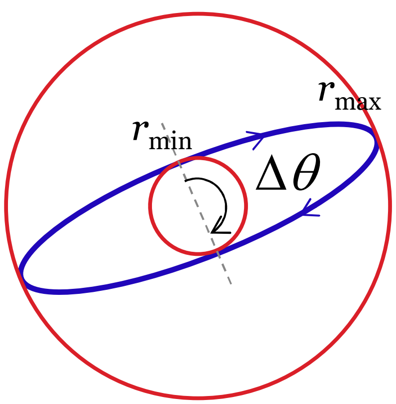
$$\left.\begin{matrix} n = 5, 3, 0, -4, -5, -7 \\ \text{or} \\ n = -\frac{3}{2}, -\frac{5}{2}, -\frac{1}{3}, -\frac{5}{7}, -\frac{7}{3} \end{matrix}\right\} \text{can be solved in terms of elliptic functions}$$

Closed Orbits

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

Comments:

2. This form of the equation is also useful in determining whether or not orbits are **closed**, i.e. if they eventually return to where it started and retrace the same path



→ Recall we showed that the orbit is symmetric about its apsides

→ So the angular change in θ in going from $r_{\min}$ to $r_{\max}$ then $r_{\max}$ back to $r_{\min}$ is

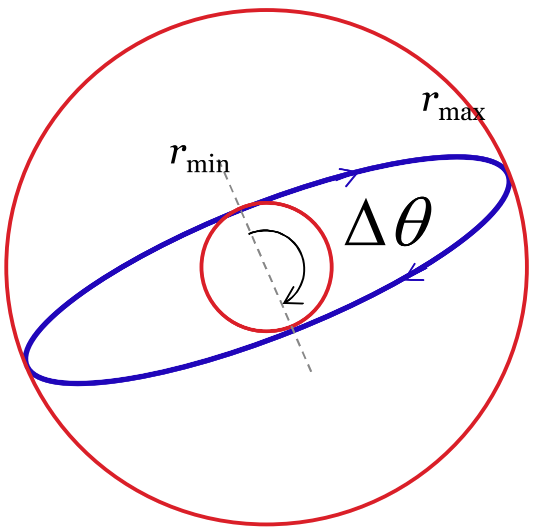
$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} l\,dr \Big/ mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

Closed Orbits: Bartard's Theorem

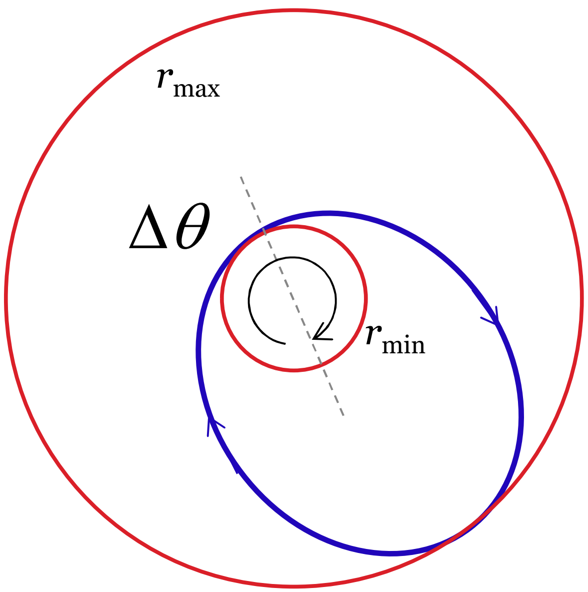
Comments:

→ If $\Delta\theta = 2\pi\left(\frac{a}{b}\right)$, where $\frac{a}{b}$ is rational, then the orbit closes after b cycles of: $r_{\min} \rightarrow r_{\max} \rightarrow r_{\min}$. And, the orbit will have gone around the center-of-force a times.

In this example, $\frac{a}{b} = \frac{1}{2}$, the orbit is a closed ellipse.



Closed Orbits: another example



Another example with $\dfrac{a}{b} = 1$: the orbit is a closed ellipse.

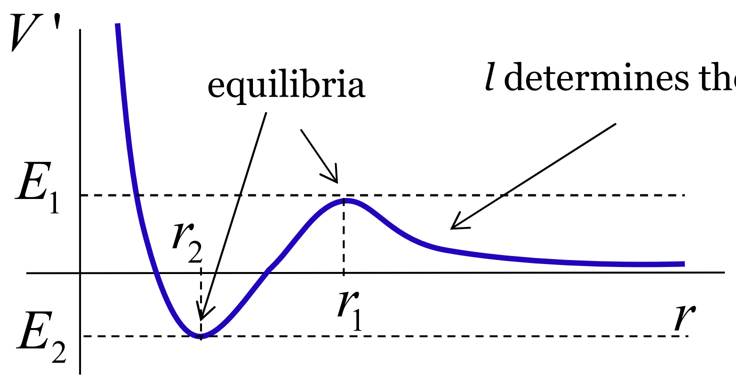
Bartard's Theorem (1873) states that only the inverse square force (n = -2) and Hooke's law (n = 1) give rise to closed orbits.

(We won't prove it but we will give a favor of it now.)

Stability of Circular Orbits

- For any *attractive* potentials, a (bounded) *circular* orbit is always possible for the right choice of E and l.

This circular orbit will occur at r values where the effective potential $V'(r)$ has its extrema (equilibria).



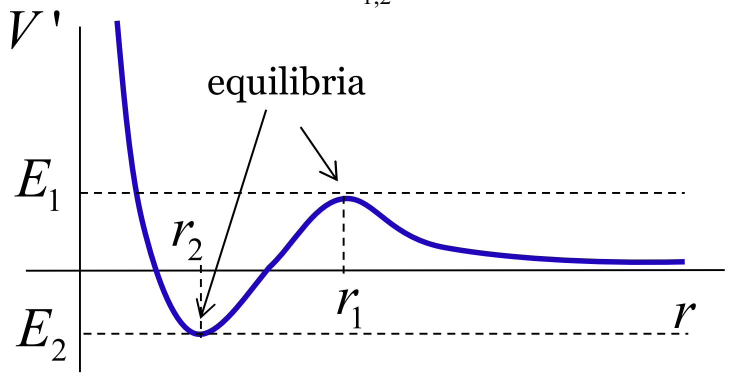
For $E = E_{1,2}$, orbit will be a circular orbit at $r = r_{1,2}$.

(Note: The $-\dfrac{l^2}{2mr^2}$ term in V' (r) is "repulsive" so that an attractive potential $V(r)$ is needed to "create" a potential well.)

Stability of Circular Orbits

- A given circular orbit is *stable* if:

$$\left.\frac{dV'}{dr}\right|_{r=r_{1,2}} = 0 \qquad \text{AND} \qquad \left.\frac{d^2V'}{dr^2}\right|_{r=r_{1,2}} > 0$$



So, for this example, the circular orbit at $r = r_2$ will be stable and the one at $r = r_1$ will not be.

- Consider a general power law attractive central force:

$$F(r) = -\frac{k}{r^n} \qquad \text{with} \qquad V(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}}$$

Stability of Circular Orbits

and the effective potential is: $V'(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}} + \frac{l^2}{2mr^2}$

- Applying the condition for stable circular orbits:

① $$\left.\frac{dV'}{dr}\right|_{r=r_0} = \frac{k}{r_0^n} - \frac{l^2}{mr_0^3} = 0 \Rightarrow \boxed{r_0^{(n-3)} = \frac{mk}{l^2}}$$

② $$\left.\frac{d^2V'}{dr^2}\right|_{r=r_0} = -\frac{nk}{r_0^{n+1}} + \frac{3l^2}{mr_0^4} > 0 \Rightarrow \boxed{-\frac{nk}{r_0^{n-3}} + \frac{3l^2}{m} > 0}$$

Substitute $r_0^{(n-3)}$ from the top into the bottom equation, we have,

$$-n\cancel{k}\left(\frac{l^2}{m\cancel{k}}\right) + \frac{3l^2}{m} > 0 \Rightarrow \boxed{(3-n)\left(\frac{l^2}{m}\right) > 0}$$

Stability of Circular Orbits

The red boxed equation implies that:

$n < 3$ is required for stable circular orbits!

- Now, we consider the situation if the orbit is slightly deviated from the stable circular orbit.

⇨ We want to analyze its oscillations about the circular orbit…

For convenience, we rescale the force law by m such that:

$$F(r) = -mg(r) = -\frac{dV}{dr}$$

so that the r equation of motion is given by:

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{dV}{dr} = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = -g(r)$$

Stability of Circular Orbits

Substituting the constant angular momentum: $l = mr^2\dot{\theta}$

$$\ddot{r} - \frac{l^2}{m^2 r^3} = -g(r)$$

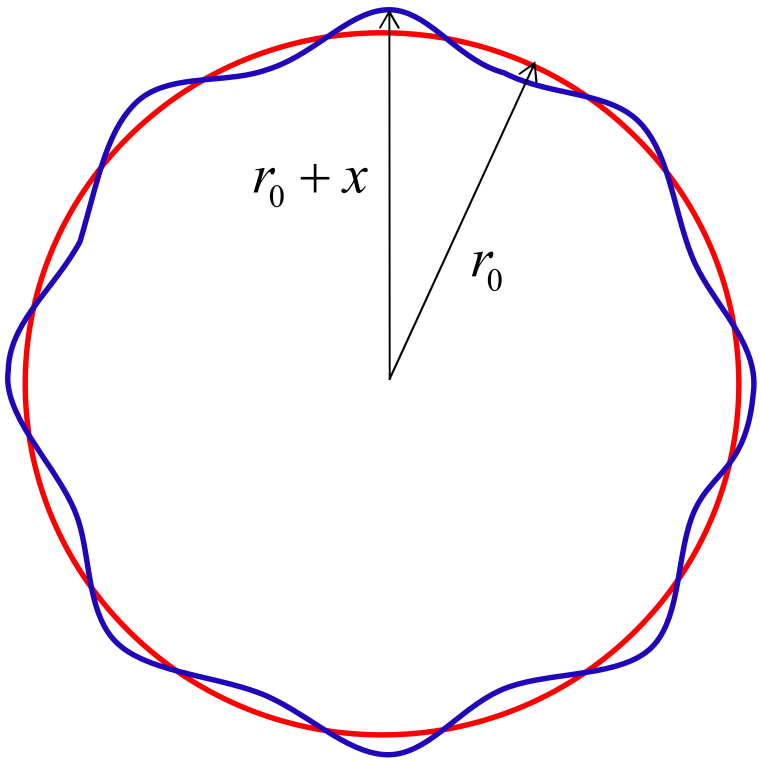
- Now, we consider the situation when the orbit was initially at $r = r_0$ and we apply a small perturbation x to it, i.e., $r \rightarrow r_0 + x$ (click)

$$\ddot{x} - \frac{l^2}{m^2 \left(r_0 + x\right)^3} = -g(r_0 + x) \qquad (note: \quad \ddot{r} = \ddot{x})$$

Under this small perturbation, we want to approximate:

① $$\frac{1}{\left(r_0 + x\right)^3} = \frac{1}{r_0^{\,3}\left(1 + \frac{x}{r_0}\right)^3} \cong \frac{1}{r_0^{\,3}}\left(1 - 3\frac{x}{r_0}\right)$$

Small Perturbations of a Circular Orbit



Stability of Circular Orbits

② $$g(r_0+x) \cong g(r_0) + \left.\frac{dg}{dr}\right|_{r=r_0} x$$ ① $$\frac{1}{(r_0+x)^3} \cong \frac{1}{r_0^{\ 3}}\left(1-3\frac{x}{r_0}\right)$$

- Putting these two approximations back into our ODE for the perturbation x,

$$\ddot{x} - \frac{l^2}{m^2(r_0+x)^3} = -g(r_0+x)$$

$$\cancel{\ddot{x}} - \frac{l^2}{m^2 r_0^{\ 3}}\left(1-3\cancel{\frac{x}{r_0}}\right) = -\left(g(r_0) + g'(r_0)x\right)$$

- Note, ON the circular orbit ($r = r_0$), we have $x = \dot{x} = \ddot{x} = 0$

$$\Rightarrow \quad \frac{l^2}{m^2 r_0^{\ 3}} = g(r_0)$$

Stability of Circular Orbits

- Putting this back into the perturbation equation, we have,

$$\ddot{x} - g(r_0)\left(\cancel{1} - 3\frac{x}{r_0}\right) = -\cancel{g(r_0)} - g'(r_0)x$$

$$\boxed{\ddot{x} + \left(\frac{3g(r_0)}{r_0} + g'(r_0)\right)x = 0}$$

- This looks like the harmonic oscillator equation with natural frequency ω

$$\ddot{x} + \omega^2 x = 0 \qquad \text{with} \qquad \omega^2 = \frac{3g(r_0)}{r_0} + g'(r_0)$$

Stability of Circular Orbits

- If $\omega^2 > 0$, this has the general oscillatory solution:

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

- If $\omega^2 < 0$, then ω is imaginary and the solution will no longer be oscillatory

$\Rightarrow$ The perturbation x will exponentially grow or decay in time

→ circular orbit will not be stable !

- So, for stability of the circular orbit, we need $\omega^2 > 0$:

$$\frac{3g(r_0)}{r_0} + g'(r_0) > 0$$

$$\frac{3}{r_0} + \frac{g'(r_0)}{g(r_0)} > 0 \quad \left(\text{we used } g(r_0) = \frac{l^2}{m^2 r_0^3} > 0\right)$$

$\Rightarrow$ $$\boxed{\frac{3}{r_0} + \frac{F'(r_0)}{F(r_0)} > 0}$$

Stability of Circular/Closed Orbits

- Again, using a power law force law: $F(r) = -kr^{-n}$

- We have,

$$\frac{3}{r_0} + \frac{nkr_0^{-(n+1)}}{-kr_0^{-n}} > 0$$

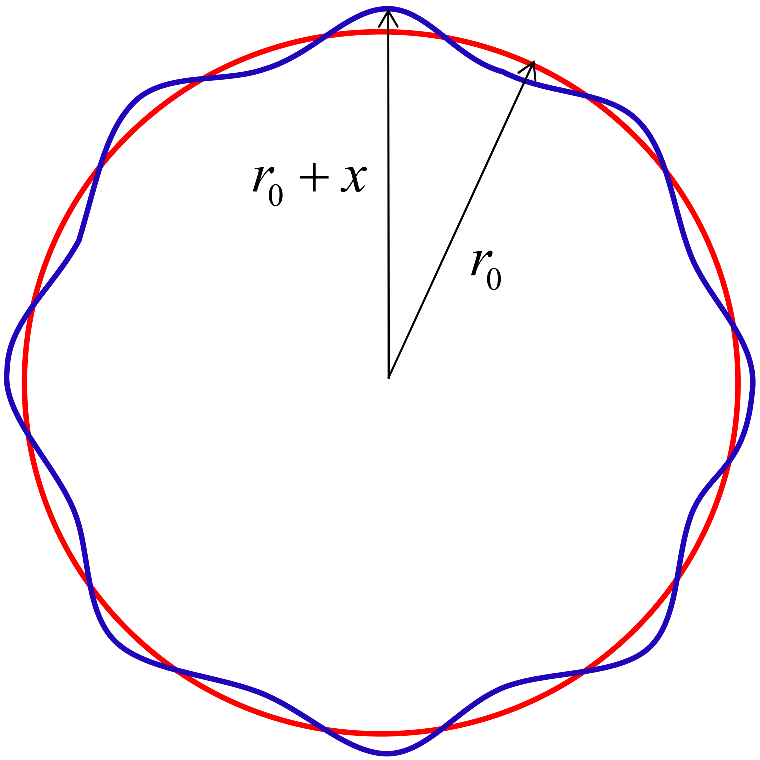
$$\frac{3}{r_0} - \frac{n}{r_0} = \frac{(3-n)}{r_0} > 0$$

- Again, we have the condition $n < 3$ needed for **stable** circular orbit.

- One step further, in order for us to have **closed** (but slightly off circular) orbits

⇨ ω , the angular speed of the deviation from the circular orbit, must be commensurate with the angular speed of the circular orbit itself, ω_0.

Small Perturbations of a Circular Orbit



$\omega \to$ one oscillation of the ripple

$\omega_0 \to$ one full cycle around the center

The blue orbit oscillate as it goes around the center of force and it closes back onto itself.

Stability of Closed Orbits

- Let consider this further. On the circular orbit, we have $\ddot{r} = 0,\ \dot{\theta} = \omega_0\ (const)$

  From the r equation of motion, we can calculate this ω_0,

$$mr_0\omega_0^2 = -F(r_0) = mg(r_0)$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 = F(r)}$$

$$\omega_0^2 = g(r_0)/r_0$$

- For closed orbits, we then need,

$$\frac{\omega}{\omega_0} = \left[\frac{\dfrac{3g(r_0)}{r_0} + g'(r_0)}{\dfrac{g(r_0)}{r_0}}\right]^{1/2} = \left[3 + \frac{r_0 g'(r_0)}{g(r_0)}\right]^{1/2} = \beta^{1/2} = \frac{p}{q}$$

p,q must be integers

Bartard's Theorem Again

Again, consider a power law force law: $F(r) = -kr^{-n}$

$$\frac{\omega}{\omega_0} \equiv \beta^{1/2} = \left[3 + \frac{r_0 g'(r_0)}{g(r_0)}\right]^{1/2} = \left[3 + \frac{r_0\left(nkr_0^{-n-1}\right)}{-kr_0^{-n}}\right]^{1/2} = \left[3-n\right]^{1/2} \stackrel{?}{=} \frac{p}{q}$$

Check: Both $n = 2, -1$ give rational solutions and they will give closed orbits !

Bartard basically repeated a similar analysis by including higher order perturbation terms to show that for all orbits (not necessary small deviations from a circular orbit) to be closed, n must be -2 or -1.

$$F(r) = -\frac{k}{r^2} \qquad F(r) = -kr$$