PHYS 705: Classical Mechanics

Introduction to Dynamical Systems
Dynamical Systems: An Introduction

A Dynamical System: A physical system described by a set of ODEs.

\[ \dot{x} = F(x, t) \quad \text{where } x \text{ is a } n \text{ dimensional vector in State Space or Phase Space} \]

A State space is the set of all possible states of a dynamical system and each state \( x \) of the system corresponds to a unique point in the state space.

Example: the state of a pendulum is uniquely defined by its angle and angular velocity, so that

\[ x = (\theta, \dot{\theta}) \]
Dynamical Systems: An Introduction

In a Dynamical System: \( \dot{x} = F(x, t) \)

\( F(x, t) \) is called a vector field (or velocity field, red arrows)

Trajectories \( x(t) \) (blue curves) will trace out curves in State Space by following the vector field (red arrows) in time.

The set of trajectories in State Space is called a Phase Portrait of the system.
Dynamical Systems: Standard Form (1st order ODE)

Any ODE can be put into the standard “vector field” form (a set of 1st order ODE),

As an example, let say we have a 2nd order ODE from the E-L equation for a generalized coordinate \( q \):

\[
\ddot{q} = f(q,t)\dot{q} + g(q,t)
\]

We can rewrite this into the standard form (a set of 1st order ODE) by introducing a new variable \( s = \dot{q} \), then we have

\[
\dot{x} = F(x,t) \quad \begin{align*}
\dot{s} &= F_1(q,s,t) \\
\dot{q} &= F_2(q,s,t)
\end{align*} = \begin{pmatrix} f(q,t)s + g(q,t) \end{pmatrix}
\]
Dynamical Systems: a simple pendulum

An example: a simple pendulum

The E-L equation gives:

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta \quad \begin{bmatrix} -\gamma \dot{\theta} \end{bmatrix} \quad \text{(if there is friction)} \]

\[ \begin{bmatrix} +A \cos(\omega t) \end{bmatrix} \quad \text{(if it is driven)} \]

Letting \( s = \dot{\theta} \), this 2\textsuperscript{nd} order ODE can then be put into two 1\textsuperscript{st} order ODEs:

\[ \dot{s} = -\frac{g}{l} \sin \theta \quad \begin{bmatrix} -\gamma s + A \cos(\omega t) \end{bmatrix} \]

\[ \dot{\theta} = s \]
Dynamical Systems: Phase Portrait of a Simple Pendulum

\[ \dot{s} = -\frac{g}{l} \sin \theta \]

\[ \dot{\theta} = s \]

The following is the phase portrait and vector field for the simple pendulum:
Dynamical Systems: Equilibria of a Simple Pendulum

An **equilibrium** for a given dynamical system: \( \dot{x} = F(x, t) \)

is given by the points \( x^* \) where the vector field vanishes, i.e.,

\[
F(x^*, t) = 0
\]

For the simple pendulum, we have:

\[
\begin{align*}
\dot{s} &= -\frac{g}{l} \sin \theta^* = 0 \\
\dot{\theta} &= s^* = 0
\end{align*}
\]

Two equilibria:

\[
\begin{align*}
\theta^* &= 0 & s^* &= 0 \\
\theta^* &= \pi & s^* &= 0
\end{align*}
\]
Dynamical Systems: Stability of an Equilibrium

To consider the stability near an equilibrium, we consider the “local” dynamics near \( x = x^* \)

Considering the simpler 1D situation first and let \( \varepsilon = x - x^* \)

Then, for \( \varepsilon \) small, we have the following variational equation for \( \varepsilon \):

\[
\dot{\varepsilon} = F'(x^*)\varepsilon + O(\varepsilon^2)
\]

\( F(x^*) = 0 \) (\( x^* \) is an equilibrium)

\[
\begin{cases}
F'(x^*) > 0 & \varepsilon \uparrow \text{ (} x^* \text{ is unstable)} \\
F'(x^*) < 0 & \varepsilon \downarrow \text{ (} x^* \text{ is stable)}
\end{cases}
\]
Dynamical Systems: Stability of an Equilibrium

For higher dimensions, the variational equation becomes:

\[ \dot{\varepsilon} = DF(x^*) \varepsilon \quad \varepsilon = x - x^*, \quad x \in \mathbb{R}^n \quad F(x^*) = 0 \]

where \( DF(x^*) \) is called the **Jacobian** matrix (a generalized “derivative”):

\[
DF(x^*) = \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{pmatrix}
\]

where \( F = (F_1, \cdots, F_n)^T \quad x = (x_1, \cdots, x_n)^T \)
Dynamical Systems: Stability of an Equilibrium

Then, the stability near the equilibrium \( x^* \) is given by the eigenvalues of \( DF(x^*) \):

- \( x^* \) is **stable** if all eigenvalues have their real part less than zero
  \[
  \text{Re} \left[ \lambda_i \left[ DF(x^*) \right] \right] < 0, \quad i = 1, \cdots, n
  \]

- \( x^* \) is **unstable** if one or more eigenvalues has a positive real part

- \( x^* \) is **neutrally stable** if the largest eigenvalue has a zero real part
Dynamical Systems: Stability of an Equilibrium

For the simple **damped** pendulum, we have the following 2D 1st order ODE (in standard form):

\[
\dot{s} = -\frac{g}{l} \sin \theta - \gamma s = F_1(\theta, s) \quad \gamma \text{ damping coefficient}
\]

\[
\dot{\theta} = s = F_2(\theta, s)
\]

\[
x^* = (s^*, \theta^*) = \begin{cases} (0, 0) \\ (0, \pi) \end{cases} \quad \text{DF}(x^*) = \begin{pmatrix} \frac{\partial F_1}{\partial s} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial s} & \frac{\partial F_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -\gamma & -\frac{g \cos \theta}{l} \\ 1 & 0 \end{pmatrix}
\]

At eq, we have

\[
\text{DF}((0, 0)) = \begin{pmatrix} -\gamma & -g/l \\ 1 & 0 \end{pmatrix} \quad \text{DF}((0, \pi)) = \begin{pmatrix} -\gamma & g/l \\ 1 & 0 \end{pmatrix}
\]
Dynamical Systems: Stability of an Equilibrium

For simplicity, we will take \( g = l = 1 \), the eigenvalues for \( \mathbf{DF}(\mathbf{x}^*) \) are:

\[
\lambda[\mathbf{DF}((0, 0))] = \left( -\frac{\gamma}{2} \pm \sqrt{1 - \left(\frac{\gamma}{2}\right)^2} \right)
\]

- Both \( \text{Re}[\lambda_{+, -}] < 0 \) \ STABLE \ (for all \ \gamma > 0)  
- Both \( \text{Re}[\lambda_{+, -}] = 0 \) \ NEUTRAL \ (for \ \gamma = 0)  

\[
\lambda[\mathbf{DF}((0, \pi))] = \left( -\frac{\gamma}{2} \pm \sqrt{1 + \left(\frac{\gamma}{2}\right)^2} \right)
\]

- \( \text{Re}[\lambda_-] < 0 \) but \( \text{Re}[\lambda_+] > 0 \) \ UNSTABLE \ (for all \ \gamma \geq 0)
Periodic Solutions (Limit Cycles) of a Simple Pendulum ($\gamma = 0$, no dissipation)

The motion can also be periodic...

Periodic solutions are in general called Limit Cycles

- neutral eq. $\theta^* = 0$
- unstable eq. $\theta^* = \pi$
Phase Portrait of a Simple Pendulum (Summary)

1. Limit Cycles:

2. Equilibria:
   - stable/neutral eq. $\theta^* = 0$
   - unstable eq. $\theta^* = \pi$
Existence & Uniqueness of Solutions and its Topological Consequence

Existence & Uniqueness statement for ODEs:

For a given ODE with a chosen IC: $\dot{x} = F(x, t), \quad x(0) = x_0$

If $F(x, t)$ is sufficiently smooth, then the initial value problem has a solution $x(t)$ on some time interval $-\tau \leq t \leq \tau$ about $t = 0$ and the solution is unique.

Uniqueness of solution implies that trajectories from a given dynamical system CANNOT cross each other.

If they did cross (let say at point $\bigcirc$), there is only ONE initial condition for the trajectory onward and going to two diff directions will violate uniqueness.
To appreciate the topological implication, consider the situation in 2D:

The blue trajectory is a limit cycle

Since trajectories cannot cross, the red trajectory must remain inside the blue loop forever!

Complex aperiodic trajectories such as chaos cannot occur in 2D!
Poincare-Bendixson Theorem

The mathematical statement of the previous geometrical observation is encapsulated in the Poincare-Bendixon Theorem.

The theorem basically stated that for any smooth dynamical systems confined within a closed and bound region in 2D plane and if there are no stable fixed points within the region, a trajectory must approach a limit cycle.

This implies typical dynamical behavior in 2D or lower is simple: equilibria, limit cycles, and possibly a combination of them.

But, NO complicated aperiodic behavior (such as chaos) is possible unless $D \geq 3$. 
The Three Body Problem & Chaos

The classic Kepler’s problem with two masses under the influence of gravity (a central force problem)

→ can be reduce to motions on a 2D plane (next chapter)
→ Bounded motions are simple circles & ellipses

However, a three-body problem will in general have 5 dof ($E$ and $L$ are conserved: $9-4=5$)

→ The system is not integrable
→ Complicated aperiodic motion including chaos is possible.
→ Initially bounded motion can be destabilized by interactions with 3$^{rd}$ body
Poincare & Geometric Analysis of ODEs

Henri Poincare (1854-1912)

→ In 1887 he won Oscar II, King of Sweden's mathematical competition by showing that the TBP cannot be solved in terms of algebraic formulas and integrals.

→ He also pioneered a geometric way in analyzing non-integrable nonlinear problems such as the TBP

(stability of eqs and periodic orbits, stable & unstable manifolds, Poincare surface of section, Poincare Recurrent Theorem, homoclinic tangles, ...)

A brief history on the development of dynamical system can be found on

http://www.scholarpedia.org/article/History_of_dynamical_systems
Dynamical Systems: Chaotic Trajectory from a Driven Damped Pendulum

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta - \gamma \dot{\theta} + A \cos(\omega t) \]

(This is a driven dissipative system with constant flow of energy into and out of the system.)

Note: This is in 3D. Time is the extra dimension which allows a trajectory to spiral back without crossing itself.
Characteristic of Chaotic Orbits

**Dynamical Description:** exponential sensitivity to initial conditions

\[ \delta x_0 \quad \Rightarrow \quad \delta x(t) \sim e^{\lambda t} \delta x_0 \]

The characteristic exponent \( \lambda \) is called the **Lyapunov exponent**.

If \( \lambda > 0 \), initial errors will grow exponentially fast and the system is **chaotic**.

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Practical implication for chaotic systems (e.g., predicting weather):

Let say we want to improve prediction time from \( \tau \) to \( 2\tau \) and \( \lambda = 9 \), how much more accurate do we need our initial estimates to be?

\[ \delta x_{2\tau} = \delta x_{\tau} \]

\[ \delta x'_0 e^{2\lambda \tau} = \delta x_0 e^{\lambda \tau} \]

\[ \frac{\delta x'_0}{\delta x_0} = e^{-2\lambda} \sim 10^{-8} \]

need improvement by 8 orders of magnitude
Characteristic of Chaotic Orbits

**Geometric Description:** stretch and fold action

- **Stretching** in a *bounded* region of space necessitates **folding**

- Trajectories still cannot cross → regions of state space must become thinner and thinner strains of “taffy” candy
- Repeating this process many many times → the resultant object will become a **fractal** (with fine spatial scaling and non-integer dimensions).

http://math.gmu.edu/~sander/movies/arnold.html
Scope of our Understanding on Physical Systems

### Number of variables

<table>
<thead>
<tr>
<th>Linear</th>
<th>Nonlinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>$n = 2$</td>
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- **Growth, decay, or equilibrium**
  - Linear oscillator
  - Mass and spring
  - RC circuit
  - RLC circuit
  - 2-body problem (Kepler, Newton)
- **Oscillations**
  - Civil engineering, structures
  - Electrical engineering

#### Continuum

- **Collective phenomena**
  - Coupled harmonic oscillators
  - Solid-state physics
  - Molecular dynamics
  - Equilibrium statistical mechanics

- **Waves and patterns**
  - Elasticity
  - Wave equations
  - Electromagnetism (Maxwell)
  - Quantum mechanics (Schrödinger, Heisenberg, Dirac)
  - Heat and diffusion
  - Acoustics
  - Viscous fluids

#### The frontier

- **Chaos**
  - Strange attractors (Lorenz)
  - 3-body problem (Poincaré)
  - Chemical kinetics
  - Iterated maps (Feigenbaum)
  - Fractals (Mandelbrot)
  - Forcible nonlinear oscillators (Levinson, Snaith)

- **Spatio-temporal complexity**
  - Coupled nonlinear oscillators
  - Lasers, nonlinear optics
  - Nonequilibrium statistical mechanics
  - Nonlinear solid-state physics (semiconductors)
  - Josephson arrays
  - Heart cell synchronization
  - Neural networks
  - Immune system
  - Ecosystems
  - Economics

**Practical uses of chaos**

- Quantum chaos?

[SH Strogatz, Non Dyn & Chaos]