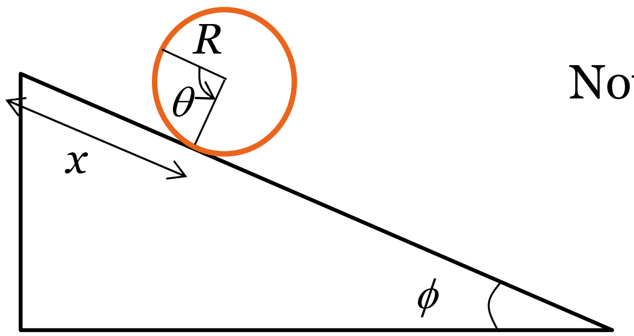
PHYS 705: Classical Mechanics

Examples: Lagrange Equations and Constraints

Hoop Rolling Down an Incline Plane



Note: • an object rolls because of friction but *static* friction does no work

• this is different from our previous case with a disk rolling on a 2D plane. This has 1 less dof

Pick the coordinates x, θ as shown. The constraint eq (rolling without slipping) is: $x - R\theta = 0$ We will solve this problem in two ways:

#1: The problem really has one "proper" generalized coordinate x and we will explicitly use the constraint equation to eliminate θ from our analysis. The EOM is simpler (1D) but we can't get an expression for the constraint force.

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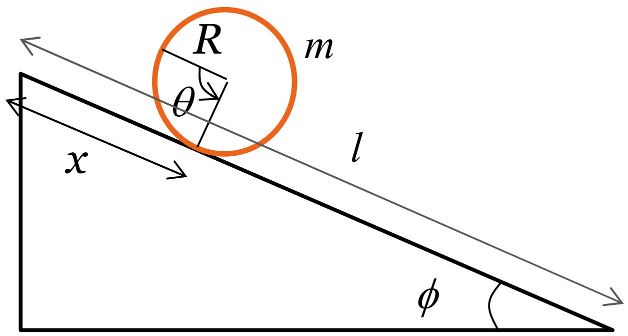

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2 \qquad I(hoop) = mR^2$$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} mR^2\dot{\theta}^2$$

$$T = m\dot{x}^2 \qquad \text{applying the constraint: } R\dot{\theta} = \dot{x}$$

Now, pick U=0 to be at where the hoop is at the bottom of the incline plane ,

we then have,

$$U = mg(l - x)\sin\phi$$

So,

$$L = m\dot{x}^2 - mg(l - x)\sin\phi$$

Lagrange Equation gives,

$$2m\ddot{x} - mg\sin\phi = 0 \quad \rightarrow \quad \ddot{x} = \frac{g\sin\phi}{2}$$

(Correct acceleration for a hoop rolling down an incline plane)

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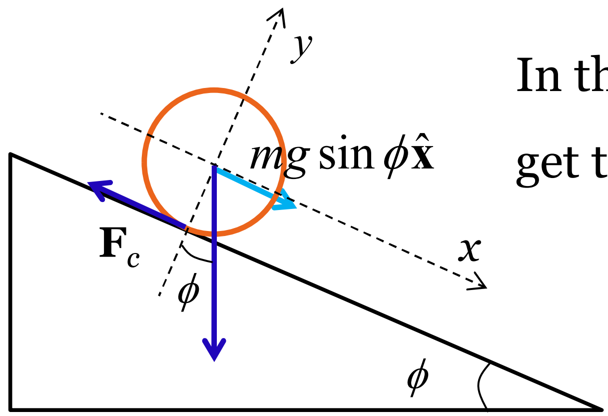

In this case, we need to go back to Newtonian mechanics to get the constraint force:

The constraint force is the static friction $\mathbf{F}_c$ needed to keep the hoop rolling without slipping.

Newton 2nd law gives, $mg\sin\phi - F_c = m\ddot{x} \quad \rightarrow \quad F_c = mg\sin\phi - m\ddot{x}$

Plug in our result for $\ddot{x}$ and we get,

$$F_c = mg\sin\phi - \frac{mg\sin\phi}{2} = \frac{mg\sin\phi}{2} \quad \Longrightarrow \quad \mathbf{F}_c = -\frac{mg\sin\phi}{2}\hat{\mathbf{x}}$$

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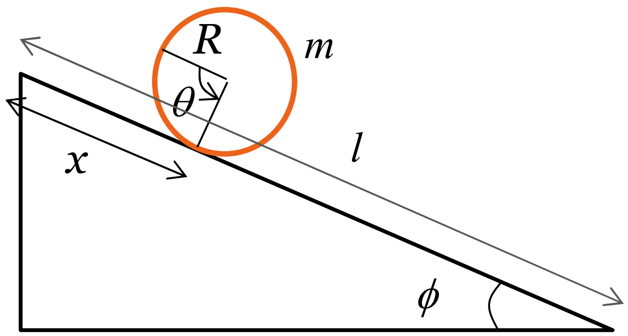

#2: Now without explicitly eliminating one of the coordinates using the constraint equation, we will use Lagrange Equation with Lagrange multipliers to get both the EOM and the magnitude of the constraint force.

Using both coordinates : x and θ

We have one holonomic constraint $g\left(x,\theta\right)=x-R\theta=0$ and we will have one Lagrange multiplier λ.

The relevant terms to be included in the Lagrange equation are:

$$\lambda\frac{\partial g}{\partial x}=\lambda \quad \text{(for } x \text{ eq)} \qquad \text{and} \qquad \lambda\frac{\partial g}{\partial \theta}=-\lambda R \quad \text{(for } \theta \text{ eq)}$$

Hoop Rolling Down an Incline Plane

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} mR^2\dot{\theta}^2$$

$$U = mg(l - x)\sin\phi$$

$$\Rightarrow \quad L = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} mR^2\dot{\theta}^2 - mg(l - x)\sin\phi$$

The EOM are:

$$\underline{x|} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0$$

$$\boxed{m\ddot{x} - mg\sin\phi = \lambda \quad (1)}$$

$$\underline{\theta|} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda\frac{\partial g}{\partial \theta} = 0$$

$$mR^2\ddot{\theta} = -\lambda R$$

$$\boxed{-mR\ddot{\theta} = \lambda \quad (2)}$$

We have three unknowns: $x, \theta,$ and λ to be solved here.

Together with the constraint equation $\boxed{x - R\theta = 0 \quad (3)}$ these system of equations can be solved. (Note: Constraint Eq is applied AFTER EOM is obtained!)

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Combining Eqs (1) and (2) by eliminating λ, we have,

$$m\ddot{x} - mg\sin\phi = -mR\ddot{\theta}$$

Now, applying the constraint using Eq (3), we have $\ddot{x} = R\ddot{\theta}$

Substituting this into the equation above, we have,

$$m\ddot{x} - mg\sin\phi = -m\ddot{x}$$

$$\boxed{\ddot{x} = \frac{g\sin\phi}{2}}$$

(same EOM for x as previously)

Now, we can substitute this back into Eq (1) to solve for λ,

$$\lambda = m\ddot{x} - mg\sin\phi = \frac{mg\sin\phi}{2} - mg\sin\phi = -\frac{mg\sin\phi}{2}$$

Hoop Rolling Down an Incline Plane

The magnitude of the force of constraint corresponding to the x-EOM is given by:

$$\left|Q_x\right| = \left|\lambda \frac{\partial g}{\partial x}\right| = \frac{mg\sin\phi}{2}$$

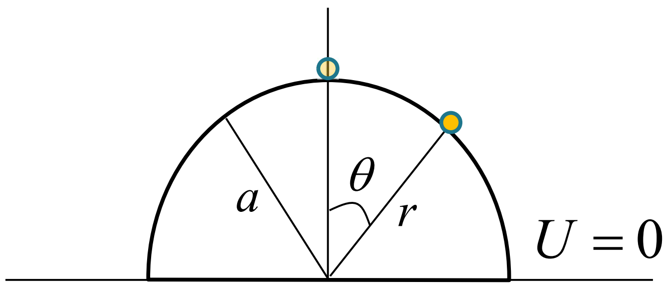
By the way, we can also get the EOM for the θ variable,

$$-mR\ddot{\theta} = \lambda = -\frac{mg\sin\phi}{2}$$

$$\ddot{\theta} = \frac{g\sin\phi}{2R}$$

Notice that there is another force of constraint (the normal force : $F_N = mg\cos\phi$). We could get that out by introducing another "improper" coordinate y that permits motion normal to the incline plane and imposing the constraint y=0.

Mass Rolling off from a Hemispheric Surface



Problem: A point mass sits on top of a smooth fixed hemisphere with radius a. Find the force of constraint and the angle at which it flies off the sphere.

Use coordinates: r and θ and constraint: $g(r,\theta) = r - a = 0$

$$T = \frac{1}{2}mv^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad \text{note: } \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$U = mgr\cos\theta$$

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

$$\lambda\frac{\partial g}{\partial r} = \lambda$$

$$\lambda\frac{\partial g}{\partial \theta} = 0$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

Mass Rolling off from a Hemispheric Surface

$\underline{r}|$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} - \lambda\frac{\partial g}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda \quad (1)$$

$\underline{\theta}|$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda\frac{\partial g}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\sin\theta = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - g\sin\theta = 0 \quad (2)$$

Inserting constraint: $r = a$ and $\dot{r} = \ddot{r} = 0$, we have

$$-ma\dot{\theta}^2 + mg\cos\theta = \lambda \quad (1')$$

$$a\ddot{\theta} - g\sin\theta = 0$$

$$\ddot{\theta} = \frac{g}{a}\sin\theta \quad (2')$$

NOTE: To find force of constraint, insert constraint conditions AFTER you have gotten the E-L equation (with the multiplier) already.

Mass Rolling off from a Hemispheric Surface

Note that $\frac{d}{dt}\left(\dot{\theta}^2\right) = 2\dot{\theta}\ddot{\theta}$

$$\ddot{\theta} = \frac{g}{a}\sin\theta$$

Substituting $\ddot{\theta}$ from Eq (2') into the above equation, we have,

$$\frac{d(\dot{\theta}^2)}{dt} = 2\dot{\theta}\left(\frac{g}{a}\sin\theta\right) = \frac{2g}{a}\sin\theta\dot{\theta} = -\frac{2g}{a}\frac{d\left(\cos\theta\right)}{dt}$$

$$d(\dot{\theta}^2) = -\frac{2g}{a}d\left(\cos\theta\right)$$

Integrating both sides, we arrive at the EOM for θ,

$$\dot{\theta}^2 = -\frac{2g}{a}\cos\theta + C \longleftarrow$$

$$\dot{\theta}^2 = \frac{2g}{a}\left(1-\cos\theta\right)$$

C is an integration constant. Assuming initial condition with $\dot{\theta}(0) = \theta(0) = 0$, we have $C = 2g/a$

Mass Rolling off from a Hemispheric Surface

$$\boxed{-ma\dot{\theta}^2 + mg\cos\theta = \lambda \quad (1')}$$

Plugging the last expression into Eq (1'), we have

$$-m\cancel{a}\left(\frac{2g}{\cancel{a}}(1-\cos\theta)\right) + mg\cos\theta = \lambda$$

$$-2mg + 2mg\cos\theta + mg\cos\theta = \lambda$$

$$\boxed{\lambda = mg(3\cos\theta - 2)}$$

(This gives the mag of the constraint force.)

The particle flies off when the constraint force = 0. By setting λ =0, we have the condition,

$$mg(3\cos\theta_c - 2) = 0$$

$$\theta_c = \cos^{-1}(2/3) \simeq 48.2^\circ$$