PHYS 705: Classical Mechanics

Derivation of Lagrange Equations from D'Alembert's Principle

D'Alembert's Principle

Using the D'Alembert's Principle and requiring the virtual displacement to be consistent with constraints, i.e, $\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$ (no virtual work for $\mathbf{f}_i$)

We can write down Newton's 2nd Law as, $\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$

Again, since the coordinates $\mathbf{r}_i$ (and the virtual variations) are not necessary independent. This does not implies, $\left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) = 0$.

However, if we can change to a set of *independent* generalized coordinates then, we can rewrite the set of equations as $\sum_j (?)_j \cdot \delta q_j = 0$ and set the independent coefficients $(?)_j = 0$ in the sum individually to zero.

Derivation of Lagrange Equations

Break $\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$ into two pieces:

1. $\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i \quad (1)$

Assume that we have a set of n=3N-K independent generalized coordinates q_j and the coordinate transformation,

$$\mathbf{r}_i = \mathbf{r}_i \left(q_1, q_2, \cdots, q_n, t \right)$$

$\mathbf{r}$ is the position vector in Cartesian Coordinates

From chain rule, we have

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

(note: $\frac{\partial \mathbf{r}_i}{\partial t} \delta t = 0$ since it is a virtual disp)

(Index convention: i goes over # particles and j over generalized coords)

Derivation of Lagrange Equations

This links the variations in $\mathbf{r}_i$ to q_j, substituting it into expression (1), we have,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = \sum_i \sum_j \left(\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \right) = \sum_j \left[\sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j$$

Defining

$$Q_j \equiv \sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

as the "generalized forces"

We can then write,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = \sum_j Q_j \, \delta q_j \qquad (1')$$

(Note: Q_j needs not have the dimensions of force but $Q_j \, \delta q_j$ must have dimensions of work.)

Derivation of Lagrange Equations

Now, we look at the second piece involving $\dot{\mathbf{p}}_i$:

2. $\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i \quad (2)$ (don't forget the "-" sign in the original Eq)

$$= \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i$$ (it is a virtual displacement so mass is constant)

$$= \sum_i m_i \ddot{\mathbf{r}}_i \cdot \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_i \sum_j \left(m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \delta q_j \qquad (2a)$$

Derivation of Lagrange Equations

Let, go backward a bit. Consider the following time derivative:

$$\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)=m_i\dot{\mathbf{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)+m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}$$

Note: This is the term from previous page

Rearranging, the term (from the previous page) can be written as,

$$m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)-m_i\mathbf{v}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)\qquad(2b)\qquad\text{where }\ \mathbf{v}_i=\frac{d\mathbf{r}_i}{dt}$$

Now, consider the blue and red terms in detail,

Derivation of Lagrange Equations

blue term: $\dfrac{\partial \mathbf{r}_i}{\partial q_j}$

Since we have $\mathbf{r}_i = \mathbf{r}_i\left(q_1, q_2, \cdots, q_n, t\right)$, applying chain rule, we have

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}$$

Taking the partial of above expression with respect to $\dot{q}_j$, we have

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$$

(note: $\mathbf{r}_i$ does not depend on $\dot{q}_j$)

red term: $\dfrac{d}{dt}\left(\dfrac{\partial \mathbf{r}_i}{\partial q_j}\right) = \dfrac{\partial}{\partial q_j}\left(\dfrac{d\mathbf{r}_i}{dt}\right) = \dfrac{\partial \mathbf{v}_i}{\partial q_j}$

(switching derivative order)

Is it ok? Check …

explicit check

$$\mathbf{r}_i = \mathbf{r}_i\left(q_1, q_2, \cdots, q_n, t\right)$$

LHS: $$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)$$

RHS: $$\frac{\partial}{\partial q_j}\left(\frac{d\mathbf{r}_i}{dt}\right) = \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}\right)$$

$$= \sum_k \frac{\partial}{\partial q_j}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\dot{q}_k + \frac{\partial}{\partial q_j}\left(\frac{\partial \mathbf{r}_i}{\partial t}\right) \quad \left(\text{note } \frac{\partial \dot{q}_k}{\partial q_j} = 0\right)$$

$$= \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)$$

Check! The two terms are the same.

Derivation of Lagrange Equations

Putting these two terms back into Eq. (2b):

$$m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)-m_i\mathbf{v}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)$$

$$m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{v}_i}{\partial\dot{q}_j}\right)-m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{v}_i}{\partial q_j}$$

With this, we finally have the following for expression (2):

$$\sum_i\dot{\mathbf{p}}_i\cdot\delta\mathbf{r}_i=\sum_i\sum_j\left(m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)\delta q_j$$

$$=\sum_i\sum_j\left[\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{v}_i}{\partial\dot{q}_j}\right)-m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{v}_i}{\partial q_j}\right]\delta q_j\qquad(2c)$$

(reminder: *i* sums over # particles and *j* sums over generalized coords)

Derivation of Lagrange Equations

We are almost there but not quite done yet. Consider taking the q_j derivative of the Kinetic Energy,

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2\right) = \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i\right)$$

$$= \frac{1}{2}\sum_i m_i \left[\left(\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right) + \left(\frac{\partial \mathbf{v}_i}{\partial q_j} \cdot \mathbf{v}_i\right)\right]$$

$$= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}$$

Similarly, we can do the same manipulations on T wrt to $\dot{q}_j$,

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2\right) = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}$$

Derivation of Lagrange Equations

Substituting these two expressions into Eq. (2c), we have:

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_j \left[\frac{d}{dt}\left(\sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

Finally, reconstructing the two terms in the D'Alembert's Principle, we have:

$$\left[\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0 \right]$$

$$\sum_j \left[Q_j - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \right] \delta q_j = 0$$

Derivation of Euler-Lagrange Equations

Now, since all the δq_j are assumed to be independent variations, the individual bracketed terms in the sum must vanish independently,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \qquad (3)$$

There are $3N$-K of these differential equations for $3N$-K q_j and the solution of these equations gives the equations of motion in terms of the generalized coords *without* explicitly needing to know the constraint forces.

Also, note the advantage of this equation as a set of scalar equations (with T) instead of the original 2nd law which is a vector equation in terms of forces.

E-L Equation for Conservative Forces

Case 1: $\mathbf{F}_i^{(a)}$ derivable from a scalar potential

$$\mathbf{F}_i^{(a)} = -\nabla_i U\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t\right) \quad \text{(note: } U \text{ not depend on velocities)}$$

$$Q_j = \sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i U \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$= -\sum_i \left(\left[\frac{\partial}{\partial x_i}\hat{\mathbf{i}} + \frac{\partial}{\partial y_i}\hat{\mathbf{j}} + \frac{\partial}{\partial z_i}\hat{\mathbf{k}} \right] U \cdot \frac{\partial}{\partial q_j}\left[x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}} \right] \right)$$

$$= -\sum_i \left(\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial U}{\partial z_i}\frac{\partial z_i}{\partial q_j} \right)$$

$$\boxed{Q_j = -\frac{\partial U}{\partial q_j}}$$

E-L Equation for Conservative Forces

Putting this expression into the RHS of Eq. (3), we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - U)}{\partial q_j} = 0$$

Notice that since U does not depends on the generalized velocity $\dot{q}_j$, we are free to subtract U from T in the first term,

$$\frac{d}{dt}\left(\frac{\partial (T - U)}{\partial \dot{q}_j}\right) - \frac{\partial (T - U)}{\partial q_j} = 0$$

E-L Equation for Conservative Forces

We now define the **Lagrangian** function $L = T - U$ and the desired

Euler-Lagrange's Equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Note: there is *no unique* choice of L which gives a particular set of equations of motion. Given $G(q, t)$ being a differentiable function of the generalized coordinate, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dG}{dt}$$

is a different Lagrangian but will result in the same EOM.

E-L Equation for Velocity Dependent Potentials

Case 2: U is velocity-dependent, i.e., $U(q_j, \dot{q}_j, t)$

In this case, we redefine the generalized force as, $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$

Now, substitute this Q_j into $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$, we then have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

E-L Equation for Velocity Dependent Potentials

Combing terms using $L = T - U$, we again have the same Lagrange's Equation,

$$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial \dot{q}_j}\right) - \frac{\partial(T-U)}{\partial q_j} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

This is the case that applies to EM forces on moving charges q with velocity $\mathbf{v}$,

$$U = q(\phi - \mathbf{A}\cdot\mathbf{v})$$ where ϕ is the scalar potential

And, $$L = \frac{1}{2}mv^2 - q(\phi - \mathbf{A}\cdot\mathbf{v})$$ and $\mathbf{A}$ is the vector potential

E-L Equation for General Forces

Case 3 (General): Applied forces CANNOT be derived from a potential

One can still write down the Lagrange's Equation in general as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

Here,

- L contains the potential from conservative forces as before and
- Q_j represents the forces **not** arising from a conservative potential

E-L Equation for Dissipative Forces

Example (dissipative friction):

$$\mathbf{F}_f = \left(-k_x v_x, -k_y v_y, -k_z v_z\right)$$

For this case, one can define the *Rayleigh's dissipation function*:

$$\Im = \frac{1}{2}\sum_i \left(k_{x_i} v_{x_i}{}^2 + k_{y_i} v_{y_i}{}^2 + k_{z_i} v_{z_i}{}^2\right)$$

Then, the friction force for the i^{th} particle can be written as,

$$\mathbf{F}_{f,i} = \left(-\frac{\partial \Im}{\partial v_{x_i}}, -\frac{\partial \Im}{\partial v_{y_i}}, -\frac{\partial \Im}{\partial v_{z_i}}\right) = -\nabla_{\mathbf{v}_i} \Im$$

E-L Equation for Dissipative Forces

Plugging this into the component of the generalized force for the force of friction, we can get,

$$Q_j = \sum_i \mathbf{F}_{f,i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \cdots = -\frac{\partial \mathfrak{I}}{\partial \dot{q}_j}$$

To see this, plug in our earlier relation : $\dfrac{\partial \mathbf{r}_i}{\partial q_j} = \dfrac{\partial \mathbf{v}_i}{\partial \dot{q}_j}$, we have

$$Q_j = \sum_i \mathbf{F}_{f,i} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}$$

$$= \sum_i -\nabla_{\mathbf{v}_i} \mathfrak{I} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = -\frac{\partial \mathfrak{I}}{\partial \dot{q}_j}$$

E-L Equation for Dissipative Forces

Then, the Lagrange' Equation for the case with dissipation becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial \Im}{\partial \dot{q}_j} = Q_j$$

- Both scalar function L and $\Im$ must be specified to get EOM.
- L will contain the potential derivable from all conservative forces as previously.

Simple Applications of the Lagrangian Formulation

A particle moving under an applied force **F** in Cartesian Coordinates :

In 3D, $\mathbf{r} = (x, y, z)$ and there will be three diff eqs for the EOMs.

The Lagrangian is given by, $L = T = \dfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

Then, the x-equation is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_x$$

(Note: $Q_x \equiv \sum_i \mathbf{F}_i \cdot \dfrac{\partial \mathbf{r}_i}{\partial x} = F_x$)

This gives,

$$\frac{d}{dt}\left(m\dot{x}\right) - 0 = F_x$$

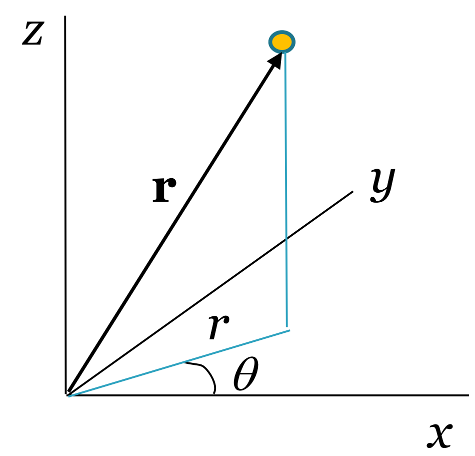
similarly for y & z →

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{F}}$$

Simple Applications of the Lagrangian Formulation

Let redo the calculation in Cylindrical Coords with the same applied force **F**:

The coordinates are: **r** = (r, θ, z)



From before, we have $T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$

Tranformation: $(x,y,z)\rightarrow(r,\theta,z)$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \rightarrow \begin{aligned} \dot{x} &= -r\sin\theta\,\dot{\theta}+\dot{r}\cos\theta \\ \dot{y} &= r\cos\theta\,\dot{\theta}+\dot{r}\sin\theta \\ \dot{z} &= \dot{z} \end{aligned}$$

Expressing T in Cylindrical Coords:

$$T=\frac{m}{2}\big(r^2\dot{\theta}^2\sin^2\theta-2r\dot{r}\dot{\theta}\sin\theta\cos\theta+\dot{r}^2\cos^2\theta$$
$$+r^2\dot{\theta}^2\cos^2\theta+2r\dot{r}\dot{\theta}\cos\theta\sin\theta+\dot{r}^2\sin^2\theta+\dot{z}^2\big)$$

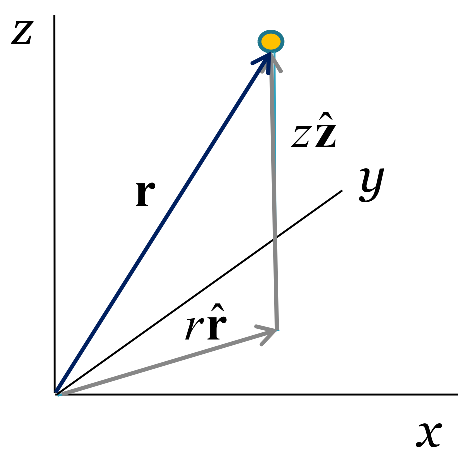
Simple Applications of the Lagrangian Formulation

Combining and canceling like-color terms, we have

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \qquad (*)$$

It is constructive to consider the following alternative way to get to this expression,

Let try to express the speed in T in cylindrical coordinates,



Start with the position vector **r**,

$$\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}$$

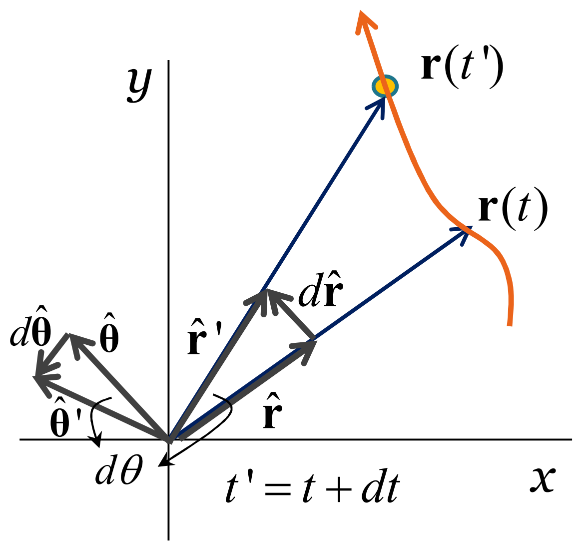
Taking the time derivative,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\hat{\mathbf{r}} + \dot{z}\hat{\mathbf{z}}$$

Note: the directional vectors $\hat{\mathbf{r}}$ change in time as the particle moves

Simple Applications of the Lagrangian Formulation

To examine on how these directional vectors changes, consider the following infinitesimal change,



Notice that,

$$\left.\begin{aligned} d\hat{\mathbf{r}} &= d\theta\hat{\boldsymbol{\theta}} \\ d\hat{\boldsymbol{\theta}} &= -d\theta\hat{\mathbf{r}} \end{aligned}\right\} \rightarrow \begin{aligned} \frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\hat{\boldsymbol{\theta}} \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\hat{\mathbf{r}} \end{aligned}$$

Back to the **v** vector, $\mathbf{v} = \frac{d\mathbf{r}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\hat{\mathbf{r}} + \dot{z}\hat{\mathbf{z}} = r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\hat{\mathbf{r}} + \dot{z}\hat{\mathbf{z}}$

So, $v^2 = \left(r\dot{\theta}\right)^2 + \dot{r}^2 + \dot{z}^2$ and $T = \frac{m}{2}v^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

Simple Applications of the Lagrangian Formulation

Now, let calculate the generalized force in cylindrical coordinates,

r:

$$Q_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} \qquad \text{Since, } \mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}\text{, we have } \frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{r}}$$

So, $\boxed{Q_r = \mathbf{F} \cdot \hat{\mathbf{r}} = F_r}$

θ:

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \longrightarrow \frac{\partial \mathbf{r}}{\partial \theta} = r\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = r\frac{\partial \theta \hat{\boldsymbol{\theta}}}{\partial \theta} = r\hat{\boldsymbol{\theta}} \quad \text{(recall previous page)}$$

So, $\boxed{Q_\theta = \mathbf{F} \cdot \left(r\hat{\boldsymbol{\theta}} \right) = rF_\theta}$ (this looks like torque)

z:

$$\boxed{Q_z = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial z} = \mathbf{F} \cdot \hat{\mathbf{z}} = F_z}$$

Simple Applications of the Lagrangian Formulation

The EOM is then given by:

Recall, $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

$$r: \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = F_r \rightarrow \boxed{m\ddot{r} - m\dot{\theta}^2 r = F_r}$$

$$\theta: \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = rF_\theta \rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = rF_\theta \quad \text{(this is } \frac{dL}{dt} = N\text{)}$$

$$\downarrow$$

$$mr^2\ddot{\theta} + m2r\dot{r}\dot{\theta} = rF_\theta \longrightarrow \boxed{m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = F_\theta}$$

$$z: \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = F_z \rightarrow \frac{d}{dt}\left(m\dot{z}\right) = F_z \rightarrow \boxed{m\ddot{z} = F_z}$$

Simple Applications of the Lagrangian Formulation

Putting the components of $\mathbf{F}$ together,

$$\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_z\hat{\mathbf{z}}$$

$$\mathbf{F} = m\left(\ddot{r} - \dot{\theta}^2 r\right)\hat{\mathbf{r}} + m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} + m\ddot{z}\hat{\mathbf{z}} \quad (*)$$

Is that the same $\mathbf{F} = m\ddot{\mathbf{r}}$ that we have gotten previously in Cartesian Coords?

Check: Recall $\mathbf{v} = \dot{\mathbf{r}} = r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\hat{\mathbf{r}} + \dot{z}\hat{\mathbf{z}}$

$$\ddot{\mathbf{r}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\hat{\mathbf{r}} + \dot{z}\hat{\mathbf{z}}\right)$$

$$\ddot{\mathbf{r}} = r\frac{d}{dt}\left(\dot{\theta}\hat{\boldsymbol{\theta}}\right) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \ddot{r}\hat{\mathbf{r}} + \ddot{z}\hat{\mathbf{z}}$$

$$= r\left(\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt} + \ddot{\theta}\hat{\boldsymbol{\theta}}\right) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \ddot{r}\hat{\mathbf{r}} + \ddot{z}\hat{\mathbf{z}}$$

Simple Applications of the Lagrangian Formulation

Using the directional vectors relations that we had earlier,

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}, \quad \frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\ddot{\mathbf{r}} = r\left(\dot{\theta}\left(-\dot{\theta}\hat{\mathbf{r}}\right) + \ddot{\theta}\hat{\boldsymbol{\theta}}\right) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\left(\dot{\theta}\hat{\boldsymbol{\theta}}\right) + \ddot{r}\hat{\mathbf{r}} + \ddot{z}\hat{\mathbf{z}}$$

Collecting all terms in the same direction,

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}$$

So the EOM in (*) on the previous page is indeed $\mathbf{F} = m\ddot{\mathbf{r}}$, i.e.,

$$\mathbf{F} = m\left[\left(\ddot{r} - \dot{\theta}^2 r\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}\right] = m\ddot{\mathbf{r}}$$