PHYS 705: Classical Mechanics

Constraints and Generalized Coordinates
Constraints

Problem Statement:

In solving mechanical problems, we start with the 2\textsuperscript{nd} law

\[
\sum_j \mathbf{F}_{ji} + \mathbf{F}_{i}^{(e)} = m_i \ddot{\mathbf{r}}_i \tag{*}
\]

In principle, one can solve for \( \mathbf{r}_i(t) \) (trajectory) for the \( i \)\textsuperscript{th} particle by specifying all the external and internal forces acting on it.

However, if \textit{constraints} are present, these external forces in general are NOT known.

Therefore, we need to understand the various constraints and know how to handle them.
Holonomic Constraints

Holonomic constraints can be expressed as a function in terms of the coordinates and time,

\[ f(r_1, r_2, \cdots; t) = 0 \]

e.g. (a rigid body) \( \Rightarrow \quad (r_i - r_j)^2 - c_{ij}^2 = 0 \)

non-holonomic examples:
- Gas in a container
- Object rolling on a rough surface without slipping... more later

More quantifiers:
- Rheonomous: depend on time explicitly
- Scleronomous: not explicitly depend on time

e.g. a bead constraints to move on a fixed vs. a moving wire
Constraints and Generalized Coordinates

Difficulties involving constraints:

1. Through \( f(r_1, r_2, \cdots; t) = 0 \), the individual coordinates \( r_i \) are no longer independent

   \[ \Rightarrow \text{eqs of motion (*) for individual particles are now coupled} \]
   (not independent)

2. Forces of constraints are not known \emph{a priori} and must be solved as additional unknowns

With \textbf{holonomic} constraints:

Prob #1 can be handled by introducing a set of “proper” (independent) Generalized Coordinates

Prob #2 can be treated with: \textbf{D’Alembert’s Principle & Lagrange’s Equations (with Lagrange multipliers)}
Generalized Coordinates

- Without constraints, a system of \( N \) particles has \( 3N \) dof.
- With \( K \) constraint equations, the \# dof reduces to \( 3N-K \).

- With holonomic constraints, one can introduce \((3N-K)\) independent (proper) generalized coordinates \( (q_1, q_2, \cdots, q_{3N-K}) \) such that:

\[
\begin{align*}
    r_1 &= r_1(q_1, q_2, \cdots, q_{3N-K}, t) \\
    \vdots \\
    r_N &= r_N(q_1, q_2, \cdots, q_{3N-K}, t)
\end{align*}
\]

\( r_i \) and \( (q_1, \cdots q_{3N-K}) \) are related by a point transformation.

- The goal is to describe the time evolution of the system in the set of \((3N-K)\) independent (proper) generalized coordinates.
Generalized Coordinates

- Generalized coordinates can be anything: angles, energy units, momentum units, or even amplitudes in the Fourier expansion of $r_i$.
- Each $q_j$ is just a number, a scalar.
- But, they must completely specify the state of a given system.
- The choice of a particular set of generalized coordinates is not unique.
- No specific rule in finding the most “suitable” (resulting in simplest EOM).
Generalized Coordinates

Example:

In Cartesian coord \( \{ \mathbf{r}_i \} \):

\[
(x_1, y_1, x_2, y_2)
\]

we have 4 dof

2 constraints:

\[
\begin{align*}
(x_1^2 + y_1^2 - l^2) &= 0 \\
((x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2) &= 0
\end{align*}
\]

So, there are only 2 indep dof...

One choice of generalized coords \( \{ q_j \} \) is:

\[
(\theta_1, \theta_2)
\]

2 indep dof

And, they are linked to the Cart. coord through:

\[
\begin{align*}
\theta_1 &= \tan^{-1}\left(\frac{x_1}{y_1}\right) \\
\theta_2 &= \tan^{-1}\left(\frac{(x_2 - x_1)}{(y_2 - y_1)}\right)
\end{align*}
\]

(Double Plane Pendulum)
Non-Holonomic Constraints

- can’t use constraint equations to eliminate dependent coordinates
- in general, solution is problem specific.

Example in book: Vertical Disk rolling without slipping on a horizontal plane

Described by 4 coordinates:
- \((x, y)\) of the contact point
- \(\theta\): orientation of disk-axis angle of disk axis with x-axis
- \(\phi\): angle of rotation of the disk
Non-Holonomic Constraints  (rolling disk exp)

Now, consider the constraints:

1. No-slip condition:
   \[ s = a\phi \quad \Rightarrow \quad v = a\dot{\phi} \]

2. Disk rolling vertically
   \[ \rightarrow \quad \mathbf{v} \perp \text{disk axis} \quad \text{see graph} \]
   \[ \Rightarrow \quad \begin{cases} \dot{x} = v \sin \theta \\ \dot{y} = -v \cos \theta \end{cases} \]
Non-Holonomic Constraints  (rolling disk exp)

Putting them together, gives the following differential equations of constraint,

\[
\begin{align*}
\frac{dx}{dt} &= a\phi \sin \theta = a \sin \theta \frac{d\phi}{dt} \\
\frac{dy}{dt} &= -a\phi \cos \theta = -a \cos \theta \frac{d\phi}{dt}
\end{align*}
\]

or

\[
\begin{align*}
dx - a \sin \theta \; d\phi &= 0 \\
dy + a \cos \theta \; d\phi &= 0
\end{align*}
\]

The point is that we can’t write this in Holonomic form:

\[
\phi - f(x, y, \theta) = 0 \quad \text{with } f \text{ being a function!} \quad \text{(hw)}
\]

Physical intuition \(\Rightarrow\) Roll the disk in a circle with radius \(R\).

Upon completion of the circle, \(x, y\) and \(\theta\) will have returned to their original values \(\Rightarrow\) but, \(\phi\) will depend on \(R\) (can’t be specified by \((x, y, \theta)\)).
How to deal with Constraints?
Principle of Virtual Work

Consider the simplest situation, a system in *equilibrium* first,

- The net force on each particle vanishes: $F_i = 0$ (note: $i$ labels the particles)

Consider an arbitrary “virtual” infinitesimal change in the coordinates, $\delta r_i$

- Virtual means that it is done with *no change in time* during which forces and constraints do *not* change.

- These virtual displacements are done *consistent* with the constraints (we will be more specific later).

Since the net force on each particle, $F_i$ is zero (equilibrium), obviously we have:

$$\sum_i F_i \cdot \delta r_i = 0$$

(virtual work)
Principle of Virtual Work

Separating the forces into applied $F_i^{(a)}$ and constraint forces $f_i$, we have:

$$F_i = F_i^{(a)} + f_i$$

Then, we have

$$\sum_i F_i^{(a)} \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0$$

Now, what do we mean by the virtual displacements $\delta r_i$ being done consistent with the constraint?
Principle of Virtual Work

For virtual displacements to be consistent with the constraints means that

→ the virtual work done by the constraint forces along the virtual displacement must be zero.

Geometric view

\[ \mathbf{f}_i \perp \delta \mathbf{r}_i \text{ or } \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \]

For \( \delta \mathbf{r}_i \) to be consistent with constraints means that the net virtual work from the forces of constraints is zero!

\[ \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \]
Principle of Virtual Work

Back to our original equation for a constrained system in equilibrium,

$$\sum_i F_i^{(a)} \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0$$

With the virtual displacements satisfying the constraints leaves us with the statement,

$$\sum_i F_i^{(a)} \cdot \delta r_i = 0$$

→ The virtual work of the applied forces must also vanish!

This is called the **Principle of Virtual Work**.
Principle of Virtual Work

\[ \sum_i F_i^{(a)} \cdot \delta r_i = 0 \]

**Note:** Since the coordinates (and the virtual variations) are not necessarily independent. They are linked through the constraint equations. The Principle of Virtual Work does **not** imply,

\[ F_i^{(a)} = 0 \quad \text{for all } i \text{ independently.} \]

The trick is now to change variables to a set of proper (independent) generalized coordinates. Then, we can rewrite the equation as,

\[ \sum_j (\text{?})_j \cdot \delta q_j = 0 \]

With \( q_j \) being independent, we can then claim: \( (?)_j = 0 \) for all \( j \). As we will see, this will give us expressions which will lead to the solution of the problem.
D’Alembert’s Principle

Now, we consider the more general case when the system is not necessary in equilibrium so that the net force on the particles is NOT zero. We continue to assume the constraints forces to be unknown a priori...

Similar to our discussion on the Principle of Virtual Work, we would like to reformulate the mechanical problem to include the constraint forces such that they “disappear” → you solve the “new” problem using only the (given) applied forces.

This is the basis for the D’Alembert’s Principle AND by additionally choosing a set of proper **generalized coordinates**, the problem can be solved and it will result in the Euler-Lagrange’s Equations.
D’Alembert’s Principle

In deriving the Principle of Virtual Work, the system was in equilibrium.

In extending it to include dynamics, we will begin with Newton’s 2\textsuperscript{nd} law,

\[ \mathbf{F}_i = \mathbf{p}_i \quad \text{or} \quad \mathbf{F}_i - \mathbf{p}_i = 0 \quad \text{for the } i\text{th particle in the system.} \]

We again consider a virtual infinitesimal displacement \( \delta \mathbf{r}_i \) consistent with the given constraint. Since we have \( \mathbf{F}_i - \mathbf{p}_i = 0 \) for all particles,

We have,

\[ \sum_i \left( \mathbf{F}_i - \mathbf{p}_i \right) \cdot \delta \mathbf{r}_i = 0 \]

Again, we separate out the applied and constraint forces, \( \mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i \)

This gives,

\[ \sum_i \left( \mathbf{F}_i^{(a)} - \mathbf{p}_i \right) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \]
D’Alembert’s Principle

Then, following a similar argument for the virtual displacement to be consistent with constraints, i.e, \( \sum f_i \cdot \delta r_i = 0 \) (no virtual work for \( f_i \))

We can write down, \( \sum (F_i^{(a)} - \dot{p}_i) \cdot \delta r_i = 0 \)

This is the D’Alembert’s Principle.

Again, since the coordinates (and the virtual variations) are not necessary independent. This does NOT implies, \( (F_i^{(a)} - \dot{p}_i) = 0 \) for the individual \( i \).

We then need to look into changing variables to a set of independent generalized coordinates so that we have \( \sum (\?)_j \cdot \delta q_j = 0 \) with the coefficients in the sum independently equal to zero, i.e., \( (\?)_j = 0 \)
Consider a particle moving in 3D with one Holonomic constraint,

\[ \mathbf{r}(t) = (x, y, z) \]

\[ \mathbf{r}(t) \]

\[ g(\mathbf{r}, t) = 0 \]

\[
\begin{align*}
\text{equation of motion:} & \quad m\ddot{\mathbf{r}} = \mathbf{F}^{(a)} + \mathbf{f} \\
\text{equation of constraint:} & \quad g(\mathbf{r}, t) = 0
\end{align*}
\]

Here,

- \( \mathbf{F}^{(a)} \) is the known applied force
- And, we model the unknown constraint force by the vector \( \mathbf{f} \).

Note: \( \mathbf{r}(t) \) has 3 unknown components + 1 constraint trajectory (red) is constraint to move in a 3-1=2 dimensional surface (blue \( g(\mathbf{r}, t) = 0 \)).
Geometric View of the D’Alembert’s Principle

- There are three unknown components to the constraint force $\mathbf{f}$. A scalar constraint does not specify the vector $\mathbf{f}$ completely.

- There are multiple choices for $\mathbf{f}$ which satisfy $g(\mathbf{r}, t)=0$ BUT there is an additional physical restriction on $\mathbf{f}$ that we should consider...

Observation: For a given $\mathbf{f}$, adding a component // to the surface will still keep the particle on the surface (satisfying $g(\mathbf{r}, t)=0$) but will result with an additional acceleration along the surface).
Geometric View of the D’Alembert’s Principle

\( \rightarrow \) A reasonable physical argument is to restrict the choice of \( \mathbf{f} \) so that:

Constraint Force \( \mathbf{f} \) needs to lay \( \perp \) to the constraint surface

Note that \( g(\mathbf{r},t) = 0 \) is the equation for the constraint surface and

\[ \nabla g(\mathbf{r},t) \perp \text{surface} \]

So, we can “parametrized” \( \mathbf{f} \) in term of \( g(\mathbf{r},t) \),

\[ \mathbf{f} = \lambda \nabla g(\mathbf{r},t) \quad \text{where } \lambda \text{ is a parameter} \]

This gives,

\[ m\ddot{\mathbf{x}} = \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r},t) \quad \begin{cases} 4 \text{ unknowns } \mathbf{r} \text{ and } \lambda \\ 4 \text{ equations} \end{cases} \]
Geometric View of the D’Alembert’s Principle

\[ m\ddot{\mathbf{r}} = F^{(a)} + \lambda \nabla g(\mathbf{r}, t) \]
\[ g(\mathbf{r}, t) = 0 \]

\begin{align*}
\text{4 unknowns } \mathbf{r} \text{ and } \lambda \\
\text{4 equations}
\end{align*}

This system is solvable but now we would like to solve the system w/o using the constraint explicitly ...

Note that \( \nabla g \) is \( \perp \) to the surface of constraint and we can project the dynamical equation onto the tangent plane of the constraint surface at \( (\mathbf{r}, t) \).

To do that, take \( \mathbf{e}_a \) and \( \mathbf{e}_b \) as two basis vectors spanning the tangent plane to the constraint surface at \( (\mathbf{r}, t) \). Dotting the above Eq to \( \mathbf{e}_a \) and \( \mathbf{e}_b \) gives two independent scalar equations,

\[ (m\ddot{\mathbf{r}} - F^{(a)}) \cdot \mathbf{e}_a = \lambda \nabla g(\mathbf{r}, t) \cdot \mathbf{e}_a = 0 \]
\[ (m\ddot{\mathbf{r}} - F^{(a)}) \cdot \mathbf{e}_b = \lambda \nabla g(\mathbf{r}, t) \cdot \mathbf{e}_b = 0 \]
Geometric View of the D’Alembert’s Principle

Together with the constraint equation itself, we then have 3 eqs for the 3 unknown components of $r$.

\[
\begin{align*}
(m\ddot{r} - F^{(a)}) \cdot e_{a,b} &= 0 \\
g(r, t) &= 0
\end{align*}
\]

So, now, in principle, we can solve for the dynamical equation (EOM), $r(t)$, without knowing the constraint forces $f$ explicitly.

\[
(m\ddot{r} - F^{(a)}) \cdot e_{a,b} = 0
\]

→ This is the D’Alembert’s Principle (for a single particle).
Geometric View of the D’Alembert’s Principle

We can generalize the argument to a system of $N$ particles with $K$ constraints (Holonomic):

$$\sum_i \left( m_i \ddot{r}_i - F_i^{(a)} \right) \cdot e_k = 0$$

$$\left( \sum_i (\dot{p}_i - F_i^{(a)}) \cdot \delta r_i = 0 \right)$$

Note: The virtual $\delta r_i$ displacements consistent with the constraints are in the tangent space spanned by the basis $\{e_k\}$

**Geometric Interpretation:**

The $K$ constraints restrict the system to a $(3N-K)$-D surface within the $3N$-D space. There are $(3N-K)$ $e_k$ vectors spanning that tangent plane to the constraint surface so that the above expression gives $(3N-K)$ equations that the problem can be solved without knowing the constraint forces explicitly.
D’Alembert’s Principle

\[ \sum_i \left( \mathbf{F}_{i}^{(a)} - \mathbf{p}_i \right) \cdot \delta \mathbf{r}_i = 0 \]

To solve for EOM using the D’Alembert’s Principle ...

We still need to look into changing variables to a set of independent generalized coordinates so that we have

\[ \sum_j (?)_j \cdot \delta q_j = 0 \]

Then, we can claim the “coefficients” \((?)_j\) in the sum to be independently equal to zero, i.e.,

\[(?)_j = 0\]
D’Alembert’s Principle

\[ \sum_i \left( F_i^{(a)} - \dot{p}_i \right) \cdot \delta r_i = 0 \]

To solve for EOM using the D’Alembert’s Principle ...

We still need to look into changing variables to a set of independent generalized coordinates so that we have

\[ \sum_j \left( ? \right)_j \cdot \delta q_j = 0 \]

The correct “coefficients” allowing us to have \( (?)_j = 0 \) will give us the Euler-Lagrange equation and the EL Eq gives an explicit expression for the EOM:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \]
Side Note: Constraint and Work

Recall that we have from the EOM: \[ m\ddot{\mathbf{r}} = F^{(a)} + \lambda \nabla g(\mathbf{r}, t) \]

Let \( F^{(a)} \) be a conservative force, i.e., \( F^{(a)} = -\nabla U(\mathbf{r}, t) \) so that

\[ m\ddot{\mathbf{r}} = -\nabla U + \lambda \nabla g \]

Dotting \( \mathbf{r} \) into both sides,

\[ m\ddot{\mathbf{r}} \cdot \mathbf{r} = \frac{d}{dt} \left( \frac{1}{2} m\mathbf{\ddot{r}}^2 \right) = \frac{dT}{dt} \quad \text{or} \quad -\nabla U \cdot \dot{\mathbf{r}} + \lambda \nabla g \cdot \dot{\mathbf{r}} \]

Consider the full-time derivative of \( g \), we have,

\[ \frac{dg}{dt} = \left( \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial g}{\partial z} \frac{dz}{dt} \right) + \frac{\partial g}{\partial t} = \left( \nabla g \cdot \dot{\mathbf{r}} \right) + \frac{\partial g}{\partial t} \]
An Aside: Constraint and Work

As the particle moves, it is constraint to stay on the $g=0$ surface,

So, \( \frac{dg}{dt} = 0 \) and, \( (\nabla g \cdot \dot{r}) = -\frac{\partial g}{\partial t} \)

Similarly, considering the full-time derivate of $U$, \( \nabla U \cdot \dot{r} = \frac{dU}{dt} - \frac{\partial U}{\partial t} \)

Putting everything together,

\[
m\ddot{r} \cdot \dot{r} = -\nabla U \cdot \dot{r} + \lambda \nabla g \cdot \dot{r}
\]

With $E=T+U$,

\[
\frac{dT}{dt} = -\frac{dU}{dt} + \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}
\]

\[
\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}
\]
An Aside: Constraint and Work

\[ \frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t} \]

So, either \( U \) or \( g \) explicitly depends on time, the total energy changes with time.

Since we typically do not consider time-dependent \( U \) potential functions, So, we can make the following assertions:

Scleronomous (\( g \) not explicitly depends on \( t \)) Holonomic Constraints:

\[ (\nabla g \cdot \dot{r}) = -\frac{\partial g}{\partial t} = 0 \quad \text{and constraint force won’t do work!} \]

Rheonomous (\( g \) explicitly depends on \( t \)) Holonomic Constraints:

\[ (\nabla g \cdot \dot{r}) = -\frac{\partial g}{\partial t} \neq 0 \quad \text{and constraint force can do work!} \]