Constraints and Generalized Coordinates
Constraints

Problem Statement:

In solving mechanical problems, we start with the 2\textsuperscript{nd} law
\[
\sum_j F_{ji} + F^{(e)}_i = m_i \ddot{r}_i \tag{*}
\]

In principle, one can solve for \( r_i(t) \) (trajectory) for the \( i \textsuperscript{th} \) particle by specifying all the external and internal forces acting on it.

However, if constraints are present, these external forces in general are NOT known.

Therefore, we need to understand the various constraints and how to handle them.
Holonomic Constraints

**Holonomic constraints** can be expressed as a function in terms of the coordinates and time,

\[ f(r_1, r_2, \cdots ; t) = 0 \]

- e.g. (a rigid body) \( \Rightarrow \) \( (r_i - r_j)^2 - c_{ij}^2 = 0 \)

- non-holonomic examples:
  - Gas in a container
  - Object rolling on a rough surface without slipping... more later

- More quantifiers:
  - **Rheonomous**: depend on time explicitly
  - **Scleronomous**: not explicitly depend on time

- e.g. a bead constraints to move on a fixed vs. a moving wire
Constraints and Generalized Coordinates

Difficulties involving constraints:

1. Through \( \bar{f}(\mathbf{r}_1, \mathbf{r}_2, \cdots; t) = 0 \), the individual coordinates \( \mathbf{r}_i \) are no longer independent

   \( \Rightarrow \) eqs of motion (*) for individual particles are now coupled (not independent)

2. Forces of constraints are not known \textit{a priori} and must be solved as additional unknowns

With holonomic constraints:

Prob #1 can be solved by introducing a set of “proper” (independent) Generalized Coordinates

Prob #2 can be treated with: D’Alembert’s Principle & Lagrange’s Equations (with Lagrange multipliers)
**Generalized Coordinates**

- Without constraints, a system of $N$ particles has $3N$ dof.
- With $K$ constraint equations, the number of dof reduces to $3N-K$.
- With holonomic constraints, one can introduce $(3N-K)$ independent (proper) generalized coordinates $(q_1, q_2, \cdots, q_{3N-K})$ such that:

  \[
  \mathbf{r}_i = \mathbf{r}_i \left( q_1, q_2, \cdots, q_{3N-K}, t \right) \\
  \vdots \\
  \mathbf{r}_N = \mathbf{r}_N \left( q_1, q_2, \cdots, q_{3N-K}, t \right)
  \]

  a point transformation

  - Generalized coordinates can be anything: angles, energy units, momentum units, or even amplitudes in the Fourier expansion of $\mathbf{r}_i$.
  - But, they must completely specify the state of a given system.
  - The choice of a particular set of generalized coordinates is not unique.
  - No specific rule in finding the most “suitable” (resulting in simplest EOM).
Generalized Coordinates

Example:

In regular Cartesian coord \( \{ r_i \} \):

\[
(x_1, y_1, x_2, y_2)
\]

4 dof

2 constraints:

\[
\begin{align*}
x_1^2 + y_1^2 - l^2 &= 0 \\
(x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 &= 0
\end{align*}
\]

In generalized coord \( \{ q_j \} \):

\[
(\theta_1, \theta_2)
\]

2 \text{ indep dof}

Coord Transformation:

\[
\theta_1 = \tan^{-1}\left(\frac{x_1}{y_1}\right)
\]

\[
\theta_2 = \tan^{-1}\left(\frac{(x_2 - x_1)}{(y_2 - y_1)}\right)
\]

(Double Plane Pendulum)

(constraints are implicitly encoded here)
Non-Holonomic Constraints

- can’t use constraint equations to eliminate dependent coordinates
- in general, solution is problem specific.

Example: Vertical Disk rolling without slipping on a horizontal plane

Described by 4 coordinates:

(x, y) of the contact point

θ: orientation of disk-axis angle of disk axis with x-axis

ϕ: angle of rotation of the disk
Non-Holonomic Constraints  (rolling disk exp)

Now, consider the no-slip condition:

1. \[ s = a\dot{\theta} \quad \Rightarrow \quad v = a\dot{\theta} \]

2. disk rolling vertically
   \[ \rightarrow \quad v \perp \text{disk axis} \quad \text{see graph} \]
   \[ \Rightarrow \quad \begin{cases} \dot{x} = v \sin \theta \\ \dot{y} = -v \cos \theta \end{cases} \]
Non-Holonomic Constraints (rolling disk exp)

Putting them together, gives the following differential equations of constraint,

\[
\begin{align*}
\frac{dx}{dt} &= a\dot{\phi}\sin \theta = a\sin \theta \frac{d\phi}{dt} \\
\frac{dy}{dt} &= -a\dot{\phi}\cos \theta = -a\cos \theta \frac{d\phi}{dt}
\end{align*}
\]

The point is that we can’t write this in Holonomic form:

\[\phi - f(x, y, \theta) = 0 \text{ with } f \text{ being a function!} \quad (hw)\]

→ Roll the disk in a circle with radius \(R\).

Upon completion of the circle, \(x, y\) and \(\theta\) will have returned to their original values → but, \(\phi\) will depend on \(R\).
How to deal with Constraints?
Principle of Virtual Work

Consider a system in equilibrium first,

- The net force on each particle vanishes: \( F_i = 0 \) (note: \( i \) labels the particles)

Consider an arbitrary “virtual” infinitesimal change in the coordinates, \( \delta r_i \)

- Virtual means that it is done with no change in time during which forces and constraints do not change.

- These changes are done consistent with the constraints (we will be more specific later).

Since all the \( F_i \) are zero (equilibrium), obviously we have \( \sum_i F_i \cdot \delta r_i = 0 \)

(virtual work)
Principle of Virtual Work

Separating the forces into applied $F_i^{(a)}$ and constraint forces $f_i$,

$$F_i = F_i^{(a)} + f_i$$

Then,

$$\sum_i F_i^{(a)} \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0$$
Principle of Virtual Work

For virtual displacements to be consistent with the constraints means that

→ the virtual work done by the constraint forces along the virtual displacement must be zero.

Alternative geometric view (more a bit later)

For $N$ particles with $K$ constraints, motion is restricted on a $(3N-K)$-D surface and the constraint forces $\mathbf{f}_i$ must be $\perp$ to that surface.

For $\delta \mathbf{r}_i$ to be consistent with constraints means that the net virtual work of the forces of constraints is zero!

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$
Principle of Virtual Work

Back to our original equation for a constrained system in equilibrium,

$$\sum_{i} F^{(a)}_i \cdot \delta r_i + \sum_{i} f_i \cdot \delta r_i = 0$$

This leaves us with the statement,

$$\sum_{i} F^{(a)}_i \cdot \delta r_i = 0$$

→ The virtual work of the applied forces must vanish!

This is called the Principle of Virtual Work.
Principle of Virtual Work

\[ \sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0 \]

**Note:** Since the coordinates (and the virtual variations) are not necessary independent. They are linked through the constraint equations. The Principle of Virtual Work does **not** implies,

\[ \mathbf{F}_i^{(a)} = 0 \quad \text{for all } i \text{ independently}. \]

The trick is now to change variables to a set of proper (independent) generalized coordinates. Then, we can write,

\[ \sum_j (\cdot) \cdot \delta q_j = 0 \]

With \( q_j \) being independent, we can then claim: \( (\cdot)_{j} = 0 \) for all \( j \). As we will see, this will give us expressions which will lead to the solution of the problem.
D’Alembert’s Principle

Now, we consider the more general case when the system is not necessary in equilibrium so that the net force on the particles is NOT zero. We continue to assume the constraints forces to be unknown \textit{a priori}...

Similar to our discussion on the Principle of Virtual Work, we would like to reformulate the mechanical problem to include the constraint forces such that they “disappear” → you solve the “new” problem using only the (given) applied forces.

This is the basis for the D’Alembert’s Principle and it will result in the Euler-Lagrange’s Equations.
D’Alembert’s Principle

In deriving the Principle of Virtual Work, the system was in equilibrium.

In extending it to include dynamics, we will begin with the 2nd law,

\[ \mathbf{F}_i = \dot{\mathbf{p}}_i \quad \text{or} \quad \mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \quad \text{for the } i^\text{th} \text{ particle in the system.} \]

We again consider a virtual infinitesimal displacement \( \delta \mathbf{r}_i \) consistent with the given constraint. Since we have \( \mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \) for all the particles,

We have,

\[ \sum_i \left( \mathbf{F}_i - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0 \]

Again, we separate out the applied and constraint forces, \( \mathbf{F}_i = \mathbf{F}^{(a)}_i + \mathbf{f}_i \)

This gives,

\[ \sum_i \left( \mathbf{F}^{(a)}_i - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \]
D’Alembert’s Principle

Then, following a similar argument for the virtual displacement to be consistent with constraints, i.e., \( \sum f_i \cdot \delta r_i = 0 \) (no virtual work for \( f_i \))

We can write down, \( \sum \left( F_i^{(a)} - \dot{p}_i \right) \cdot \delta r_i = 0 \)

This is the D’Alembert’s Principle.

Again, since the coordinates (and the virtual variations) are not necessary independent. This does NOT implies, \( \left( F_i^{(a)} - \dot{p}_i \right) = 0 \) for the individual \( i \).

We then need to look into changing variables to a set of independent generalized coordinates so that we have \( \sum (?)_j \cdot \delta q_j = 0 \) with the coefficients in the sum independently equal to zero, i.e., \( (?)_j = 0 \)
Geometric View of the D’Alembert’s Principle

Consider a particle moving in 3D with one Holonomic constraint,

\[
\begin{align*}
\text{equation of motion:} & \quad m\ddot{\mathbf{x}} = \mathbf{F}^{(a)} + \mathbf{f} \\
\text{equation of constraint:} & \quad g(\mathbf{x}, t) = 0
\end{align*}
\]

Here,
- \( \mathbf{F}^{(a)} \) is the known applied force
- And, we model the unknown constraint force by the vector \( \mathbf{f} \).

\( \text{trajectory (red)} \) is constraint to move in a 3-1=2 dimensional surface (blue).

Note: \( \mathbf{r}(t) \) has 3 unknown components + 1 constraint
Geometric View of the D’Alembert’s Principle

- There are three unknown components to the constraint force $f$. A scalar constraint does not specify the vector $f$ completely.

- There are multiple choices for $f$ which satisfy $g(x, t)=0$ BUT there is an additional physical restriction on $f$ that we should consider...

Observation: For a given $f$, adding a component // to the surface will keep the particle on the surface (satisfying $g(x, t)=0$) but will result with an additional acceleration along the surface).
Geometric View of the D’Alembert’s Principle

→ A reasonable physical argument is to restrict the choice of $\mathbf{f}$ so that:

Constraint Force $\mathbf{f}$ needs to lay $\perp$ to the constraint surface

Note that $g(x,t) = 0$ is the equation for the constraint surface, so that

$$\nabla g(x,t) \perp \text{surface}$$

So, the $\perp$ condition can be stated as,

$$\mathbf{f} = \lambda \nabla g(x,t) \quad \text{where } \lambda \text{ can depend on } t$$

This gives,

$$m\ddot{x} = F^{(a)} + \lambda \nabla g(x,t) \begin{cases} 4 \text{ unknowns } \mathbf{x} \text{ and } \lambda \\ 4 \text{ equations} \end{cases}$$
Geometric View of the D’Alembert’s Principle

\[
\begin{align*}
    m \ddot{x} &= F^{(a)} + \lambda \nabla g(x, t) \\
g(x, t) &= 0
\end{align*}
\]

This system is solvable but now we might would like to solve the system w/o knowing the constraint forces explicitly ...

Note that \( \nabla g \) is \( \perp \) to the surface of constraint and we can project the dynamics onto the tangent plane to the constraint surface at \((x, t)\):

This gives two independent scalar equations,

\[
\begin{align*}
    (m \ddot{x} - F^{(a)}) \cdot e_a &= \lambda \nabla g(x, t) \cdot e_a = 0 \\
    (m \ddot{x} - F^{(a)}) \cdot e_b &= \lambda \nabla g(x, t) \cdot e_b = 0
\end{align*}
\]

where \( e_a \) and \( e_b \) are two basis vectors spanning the tangent plane to the constraint surface at \((x, t)\).
Geometric View of the D’Alembert’s Principle

Together with the constraint equation itself, we then have 3 eqs for the 3 unknown components of $x$.

\[
\begin{align*}
(m\ddot{x} - F^{(a)}) \cdot e_{a,b} &= 0 \\
g(x, t) &= 0
\end{align*}
\]

3 unknowns $x$
3 equations

So, now, in principle, we can solve for the dynamical equation (EOM), $x(t)$, without knowing the constraint forces $f$ explicitly.

\[
(m\ddot{x} - F^{(a)}) \cdot e_{a,b} = 0
\]

→ This is the D’Alembert’s Principle (for a single particle).
Geometric View of the D’Alembert’s Principle

We can generalize the argument to a system of $N$ particles with $K$ constraints (Holonomic):

$$
\sum_{i} \left( m_i \ddot{x}_i - F_i^{(a)} \right) \cdot e_k = 0
$$

$$
\left( \sum_{i} \left( \dot{p}_i - F_i^{(a)} \right) \cdot \delta r_i = 0 \right)
$$

**Note:** The virtual $\delta r_i$ displacements consistent with the constraints are in the tangent space spanned by the basis $\{e_k\}$.

**Geometric Interpretation:**

The $K$ constraints restrict the system to a $(3N-K)$-D surface within the full $3N$-D space. There are $(3N-K) e_k$ vectors spanning that tangent plane to the constraint surface so that the above expression gives $(3N-K)$ equations that problem can be solved without knowing the constraint forces explicitly.
D’Alembert’s Principle

\[ \sum_i \left( F_i^{(a)} - \dot{p}_i \right) \cdot \delta r_i = 0 \]

To solve for EOM using the D’Alembert’s Principle ...

We need to look into changing variables to a set of independent generalized coordinates so that we have

\[ \sum_j (?)_j \cdot \delta q_j = 0 \]

Then, we can claim the coefficients \( (?)_j \) in the sum to be independently equal to zero and the Euler-Lagrange equation will give an explicit expression for the EOM as:

\[ (?)_j = 0 \]
An Aside: Constraint and Work

Recall that we have from the EOM: \[ m\ddot{x} = F^{(a)} + \lambda \nabla g(x,t) \]

Let \( F^{(a)} \) be a conservative force, i.e., \( F^{(a)} = -\nabla U(x,t) \) so that

\[ m\ddot{x} = -\nabla U + \lambda \nabla g \]

Dotting \( \dot{x} \) into both sides,

\[ m\ddot{x} \cdot \dot{x} = \frac{d}{dt} \left( \frac{1}{2} m\dot{x}^2 \right) = \frac{dT}{dt} = -\nabla U \cdot \dot{x} + \lambda \nabla g \cdot \dot{x} \]

Consider the last term, from chain rule, we have,

\[ \frac{dg}{dt} = \left( \frac{\partial g}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial t} \right) + \frac{\partial g}{\partial t} = (\nabla g \cdot \dot{x}) + \frac{\partial g}{\partial t} \]
An Aside: Constraint and Work

As the particle moves, it is constraint to stay on the \( g=0 \) surface,

\[
\frac{dg}{dt} = 0 \quad \text{and,} \quad (\nabla g \cdot \dot{x}) = -\frac{\partial g}{\partial t}
\]

Similarly, from chain rule, we can write,

\[
\nabla U \cdot \dot{x} = \frac{dU}{dt} - \frac{\partial U}{\partial t}
\]

Putting everything together,

\[
m\ddot{x} \cdot \dot{x} = -\nabla U \cdot \dot{x} + \lambda \nabla g \cdot \dot{x}
\]

With \( E=T+U \),

\[
\frac{dT}{dt} = -\frac{dU}{dt} + \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}
\]

\[
\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}
\]
An Aside: Constraint and Work

\[
\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}
\]

So, the total energy changes iff either \( U \) or \( g \) \textit{explicitly} depends on time.

Since we typically do not consider time-dependent \( U \) potential functions, So, we can make the following assertions:

Scleronomous (\( g \) not explicitly depends on \( t \)) Holonomic Constraints:

\[
(\nabla g \cdot \dot{x}) = -\frac{\partial g}{\partial t} = 0 \quad \text{and constraint force won’t do work!}
\]

Rheonomous (\( g \) explicitly depends on \( t \)) Holonomic Constraints:

\[
(\nabla g \cdot \dot{x}) = -\frac{\partial g}{\partial t} \neq 0 \quad \text{and constraint force can do work!}
\]