PHYS 705: Classical Mechanics

- Final Exam on Dec. 14th
  - Link will be active at 8am in the morning on December 14th
  - Solution of the exam in a single pdf file must arrive in my inbox by 8am on December 15th

Tensor Index Notations

https://web.iitd.ac.in/~pmvs/courses/mcl702/notation.pdf

https://www.continuummechanics.org/tensornotationbasic.html

Hamilton's Principle

Now, back to mechanics and we will define the Action as,

$$I = \int_1^2 L\left(q_i, \dot{q}_i, t\right) dt$$

where $L = T - U$ is the Lagrangian of the system

Configuration Space: The space of the q_i's. Each point gives the full configuration of the system. The motion of the system is a path through this configuration space. (This is not necessary the real space.)

(Note: Here, we don't require all the generalized coordinates q_i to be necessarily independent (proper). The q_i's can be linked through constraints.)

Hamilton's Principle: The motion of a system from t_1 to t_2 is such that the action evaluated along the actual path is stationary.

Lagrange Equation of Motion

So, we apply our variational calculus results to the action integral.

We also further assume that the system is **monogenic**, i.e., all forces except forces of constraint are derivable from a potential function $U\left(q_i, \dot{q}_i, t\right)$ which can be a function of $q_i, \dot{q}_i,$ and t

The resultant equation is the Lagrange Equation of Motion with *N* generalized coordinates (not necessary proper) and *M* Holonomic constraints:

$$\begin{cases} \dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \displaystyle\sum_{k=1}^{M} \lambda_k(t)\dfrac{\partial g_k}{\partial q_i} = 0 \\ g_k(q_i;t) = 0 \end{cases} \qquad \begin{array}{c} i = 1, 2, \cdots, N \\ k = 1, 2, \cdots, M \end{array}$$

Forces of Constraint

Comments:

1. The Lagrange EOM can be formally written as:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_{k=1}^{M} \lambda_k(t) \frac{\partial g_k}{\partial q_i} = Q_i \qquad i = 1, 2, \cdots, N$$

where the Q_i are the generalized forces which give the magnitudes of the forces needed to produce the individual constraints.

- the generalized coordinates q_i are NOT necessary independent and they are linked through constraints.

- since the choice of the sign for λ_k is arbitrary, the direction of the forces of constraint forces cannot be determined.

Proper Generalized Coordinates

Comments:

2. If one chooses a set of "proper" (*independent*) generalized coordinates in which the (*N*-*M*) q_j's are no longer linked through the constraints, then the Lagrange EOM reduces to:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \qquad j = 1, 2, \cdots, N - M$$

- In practice, one typically will explicitly use the constraint equations to reduce the number of variables to the (*N*-*M*) proper set of generalized coordinates.
- However, one CAN'T solve for the forces of constraint here

Conservation and Symmetry

- Starting with a chosen a set of "proper" generalized coordinates q_j's

So, from Hamilton's Principle, the stationarity of the Action *I* gives the homogeneous EL equations:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

Conservation and Symmetry

There is a very strong link between Symmetry and Conservation Theorems:

- Symmetry in time ⟺ conservation of energy.
- More generally, symmetry ⟺ conserved quantity
- We will show that this holds for conservation of linear & angular momentum.
- New Terminology:

generalized momentum, canonical momentum, conjugate momentum $\equiv$

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

cyclic coordinate $\equiv$ one that doesn't appear in L, *i.e.*, $\dfrac{\partial L}{\partial q_j} = 0$

Energy Conservation and Time Invariance

Note the distinction:

$f(q,\dot{q},t)$ is time invariant $\neq$ $f(q,\dot{q},t)$ is a constant in time

$$\frac{\partial f}{\partial t} = 0 \qquad\qquad \frac{df}{dt} = 0$$

⇩ functional form of f does NOT change with a time shift:

$$t \rightarrow t + \Delta$$

f can still depends on time implicitly thru $(q,\dot{q})$

⇩ f has a constant value in time

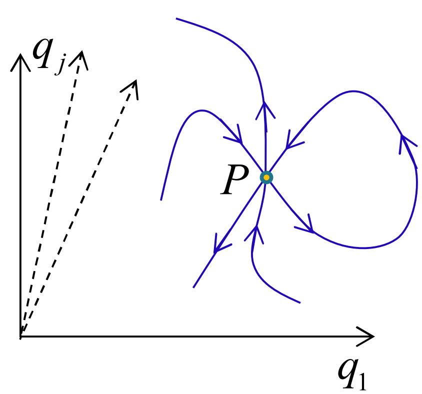
Configuration Space vs. Phase Space

① A given point in configuration space $\left(q_1, \cdots, q_n\right)$ prescribes fully the "configuration" of the system at a given time *t*.

→ However, the specification of a point in this space does NOT specify the *time evolution* of the system completely !

(a unique soln for a *n*-dim 2nd – order ODE needs 2*n* ICs)

→ Many different paths can go thru a given point in config space



← Different paths crossing *P* will have the same set of $\left\{q_j\right\}_1^n$ but diff $\left\{\dot{q}_j\right\}_1^n$

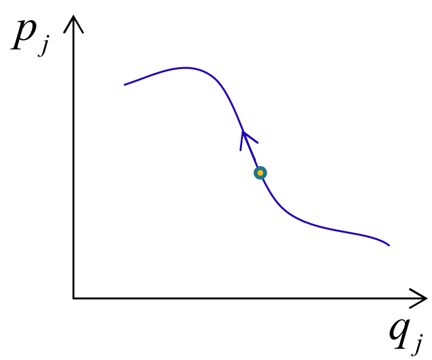
Configuration Space vs. Phase Space

② To specify the state AND time evolution of a system uniquely at a given time, one needs to specify BOTH $\{q_j\}$ AND $\{\dot{q}_j\}$ or equivalently,

$$\{q_j, p_j\}$$

→ The 2n-dim space where both $\{q_j\}$ and $\{p_j\}$ are independent variables is called *phase space*.

→ Thru any given point in phase space, there can only be ONE unique path !



GOAL: to find the EOM that applies to points in phase space.

Hamiltonian Formulation

- Instead of using the Lagrangian, $L = L\left(q_j, \dot{q}_j, t\right)$, we will introduce a new function that depends on q_j, p_j, and t: $H = H\left(q_j, p_j, t\right)$

- This new function is called the Hamiltonian and it is defined by:

$$H = \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L$$

(Einstein' s Convention: Repeated indices are summed)

- Plugging in the definition for the generalized momenta: $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$

$$H = p_j \dot{q}_j - L \qquad (sum)$$

→ One can think of this as a coordinate transformation from $\left(q_j, \dot{q}_j\right)$ to $\left(q_j, p_j\right)$

To be more specific, $H\left(q, p\right)$ is the Legendre Transform of $L\left(q, \dot{q}\right)$.

→ H is defined "similarly" to the Jacobi (energy) function h BUT h is a function of $\left(q_j, \dot{q}_j\right)$ and H is a function of $\left(q_j, p_j\right)$.

Hamiltonian Formulation

Summary of Steps:

1. Pick a proper set of q_j and form the Lagrangian L
2. Obtain the conjugate momenta by calculating $p_j \equiv \dfrac{\partial L}{\partial \dot{q}_j}$
3. Form $H = p_j \dot{q}_j - L$
4. Eliminate $\dot{q}_j$ from H using the inverse of $p_j \equiv \dfrac{\partial L}{\partial \dot{q}_j}$ so as to have

$$H = H(q_j, p_j, t)$$

5. Apply the Hamilton's Equation of Motion:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \quad and \quad \frac{\partial H}{\partial p_j} = \dot{q}_j$$

As you will see, this formulation does not necessary simplify practical calculations but it forms a theoretical bridge to QM and SM.

Symmetry and Conservation Theorem Again

- Other cyclic coordinates:

Lagrangian Formalism

If q_n is cyclic, the Lagrangian is $L = L(q_1, \cdots, q_{n-1}, \quad , \dot{q}_1, \cdots, \dot{q}_n, t)$

(q_n missing; $\dot{q}_n$ is still here)

The n-th EOM is then given by:

$$\frac{\partial L}{\partial \dot{q}_n} = const$$

(The LHS is usually a complicated function of q_j & $\dot{q}_j$ and much efforts still needed to solve for $q_n(t)$.)

Hamiltonian

In contrast, if q_n is cyclic in the Hamiltonian Formalism,

$$\frac{\partial H}{\partial q_n} = 0 = -\dot{p}_n \quad \Longrightarrow \quad p_n = const \ \text{(conserved)}$$

(This equation is immediately "solved" from IC → p_n is gone from the prob.)

(The Hamiltonian Formalism has a much nicer structure for cyclic coords)

Canonical Transformation

In phase space, a Canonical Transformation

$$Q_j = Q_j(q,p,t) \quad \text{and} \quad P_j = P_j(q,p,t)$$

is defined with the condition that there exist a transformed Hamiltonian $K(Q,P,t)$ with which the Hamilton's equations are satisfied:

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} \quad and \quad \dot{P}_j = -\frac{\partial K}{\partial Q_j}$$

The form of the EOM must be *invariant* in the new coordinates (Q,P).

$$Q_j = Q_j(q,p,t)$$
$$P_j = P_j(q,p,t)$$

Canonical Transformation: 4 Types

$$p_j \dot{q}_j - H(q,p,t) = P_j \dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}(old, new, t)$$

Type 1:

$F = F_1(q,Q,t)$

$$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t) \quad P_j = -\frac{\partial F_1}{\partial Q_j}(q,Q,t) \quad K = H + \frac{\partial F_1}{\partial t}$$

dep var

ind var

Type 2:

$F = F_2(q,P,t) - Q_i P_i$

$$p_j = \frac{\partial F_2}{\partial q_j}(q,P,t) \quad Q_j = \frac{\partial F_2}{\partial P_j}(q,P,t) \quad K = H + \frac{\partial F_2}{\partial t}$$

Type 3:

$F = F_3(p,Q,t) + q_i p_i$

$$q_j = -\frac{\partial F_3}{\partial p_j}(p,Q,t) \quad P_j = -\frac{\partial F_3}{\partial Q_j}(p,Q,t) \quad K = H + \frac{\partial F_3}{\partial t}$$

Type 4:

$F = F_4(p,P,t) + q_i p_i - Q_i P_i$

$$q_j = -\frac{\partial F_4}{\partial p_j}(p,P,t) \quad Q_j = \frac{\partial F_4}{\partial P_j}(p,P,t) \quad K = H + \frac{\partial F_4}{\partial t}$$

"Symplectic" Approach & Poisson Bracket

If a coordinate transformation $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ is canonical, then the Hamiltonian equation must be satisfied in both the original and the new set of coordinates:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}} \qquad \dot{\boldsymbol{\zeta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\zeta}}$$

With any coordinate transformation $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$, one can define the Jacobian Matrix **M**

$$M_{jk} = \frac{\partial \zeta_j}{\partial \eta_k}, \quad j,k = 1,\cdots,2n$$

If $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ is canonical, then it must satisfy the following condition:

$$\mathbf{MJM}^T = \mathbf{J}$$

Poisson Bracket

For any two function $u(q,p)$ and $v(q,p)$ depending on q and p the Poisson Bracket is defined as:

$$[u,v]_{q,p} \equiv \left(\frac{\partial u}{\partial q_j}\right)\left(\frac{\partial v}{\partial p_j}\right) - \left(\frac{\partial u}{\partial p_j}\right)\left(\frac{\partial v}{\partial q_j}\right)$$

(E's sum rule for n dof)

PB is analogous to the Commutator in QM:

$$\frac{1}{i\hbar}[\![u,v]\!] \equiv \frac{1}{i\hbar}(uv - vu)$$

where u and v are two QM operators

Poisson Bracket & Dynamics

For a given dynamical quantity $u(t)$, one can in general write down its time evolution as,

$$\dot{u} = \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

If $u = q_j \ or \ p_j$, we get back the Hamilton's Equations:

$$\dot{q}_j = [q_j, H] = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = [p_j, H] = -\frac{\partial H}{\partial q_j}$$

If u does not explicitly not depend on time explicitly, i.e., $\frac{\partial u}{\partial t} = 0$,

$$[u, H] = 0 \iff \frac{du}{dt} = 0$$

(u is a constant of motion)

Poisson Bracket & Dynamics

One can also formally write down the time evolution of $u(t)$ as a series solution in terms of the Poisson brackets evaluated at $t = 0$!

$$u(t) = u(0) + t\left[u, H\right]_0 + \frac{t^2}{2!}\left[\left[u, H\right], H\right]_0 + \cdots$$

The Hamiltonian is the **generator** of the system's motion in time !

This has a direct correspondence to the QM interpretation of H:

→ The above Taylor's expansion can be written as an "operator" eq:

$$u(t) = e^{\hat{H}t} u(0)$$

where $\hat{H}\, u(0) \equiv \left[u, H\right]\Big|_0$

$\Longleftrightarrow$

$$\left|u(t)\right\rangle = e^{iHt/\hbar} \left|u(0)\right\rangle$$

↑ (QM propagator)

Hamilton-Jacobi Equation

The Hamilton-Jacobi equation represents a very general method in solving mechanical problems.

To start…

Let say we are able to find a canonical transformation taking our $2n$ phase space variables (q_i, p_i) directly to $2n$ *constants of motion*, (β_i, α_i) i.e.,

$$Q_i = \beta_i \qquad\qquad P_i = \alpha_i$$

One sufficient condition to ensure that our new variables are constant in time is that the transformed Hamiltonian K shall be identically zero.

Hamilton-Jacobi Equation

Then, one can formally the HJ equation as:

$$H\left(q_1,\cdots,q_n;\frac{\partial F_2}{\partial q_1},\cdots,\frac{\partial F_2}{\partial q_n};t\right)+\frac{\partial F_2}{\partial t}=0$$

note that $\frac{\partial F}{\partial t}=\frac{\partial F_2}{\partial t}$ since

$F(q,P,t)=F_2(q,P,t)-Q_iP_i$

$$p_i=\frac{\partial F_2(q,\alpha,t)}{\partial q_i}$$

Notes: -Since $P_i=\alpha_i$ are constants, the HJ equation constitutes a partial differential equation of (n+1) independent variables: $(q_1,\cdots,q_n,t)$

- We used the fact that the new P's and Q's are constants but we have not specified in how to determine them yet.
- It is customary to denote the solution F_2 by S and called it the Hamilton's Principal Function.

Hamilton-Jacobi Equation

Suppose we are able to find a solution to this 1st order partial differential equation in (*n*+1) variables…

$$F_2 \equiv S = S\left(q_1, \cdots, q_n, \alpha_1, \cdots, \alpha_n; t\right)$$

where we have explicitly write out the constant new momenta: $P_i = \alpha_i$

Recall that for a Type 2 generating function, we have the following partial derivatives describing the canonical transformation:

$$p_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial q_i} \quad (T1) \qquad \left(p_i = \frac{\partial S}{\partial q_i}\right)$$

$$Q_i = \beta_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial \alpha_i} \quad (T2) \qquad \left(Q_i = \frac{\partial S}{\partial P_i}\right)$$

Hamilton-Jacobi Equation

After explicitly taking the partial derivative on the RHS of Eq. (T1)

$$p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i}$$

and evaluating them at the *initial* time t_0, we will have *n* equations that one can invert to solve for the *n* unknown constants α_i in terms of the *initial conditions* $(q_{0,} p_0, t_0)$, i.e.,

$$\alpha_i = \alpha_i(q_{0,} p_0, t_0)$$

Similarly, by explicitly evaluating the partial derivatives on the RHS of Eq. (T2) at time t_0 , we obtain the other *n* constants of motion β_i

$$Q_i = \beta_i = \left.\frac{\partial S(q,\alpha,t)}{\partial \alpha_i}\right|_{q=q_0, t=t_0}$$

Hamilton-Jacobi Equation

With all $2n$ constants of motion α_i, β_i solved, we can now use Eq. (T2) again to solve for q_i in terms of the α_i, β_i at a later time *t*.

$$\boxed{q_i = q_i(\alpha, \beta, t)} \qquad \left(\beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}\right)$$

Then, with α_i, β_i, and q_i known, we can use Eq. (T1) again to evaluate p_i in terms of α_i, β_i at a later time *t*.

$$\boxed{p_i = p_i(\alpha, \beta, t)} \qquad \left(p_i = \frac{\partial S(q(\alpha, \beta, t), \alpha, t)}{\partial q_i}\right)$$

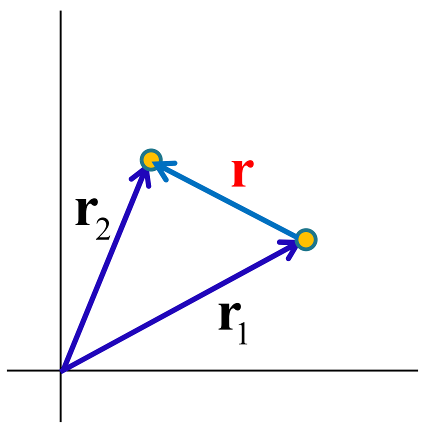
➡ The two boxed equations constitute the desired complete solutions of the Hamilton equations of motion.

Two-Body Central Force Problem

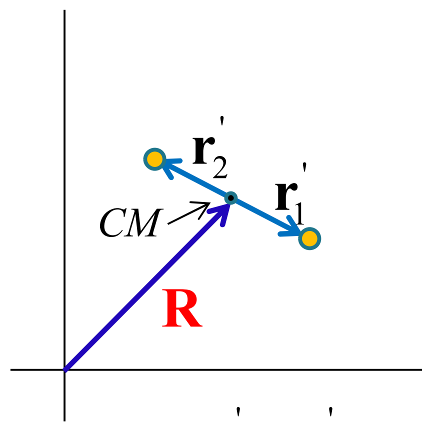
Set up of the general problem: 2 masses interact via forces directed along the line that connects them (central force): strong form of 3rd law

First Step: Central force problems can be reduced to an effective 1-body problem:

Change to generalized coordinates: CM position $\mathbf{R}$ and relative position $\mathbf{r}$,



$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$



$$\mathbf{r} = \mathbf{r}_2^{'} - \mathbf{r}_1^{'} \qquad \mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1^{'}$$

$$\mathbf{r}_2 = \mathbf{R} + \mathbf{r}_2^{'}$$

Instead of using $\mathbf{r}_1$ & $\mathbf{r}_2$, use $\mathbf{R}$ & $\mathbf{r}$ (CM & relative position)

Two-Body Central Force Problem

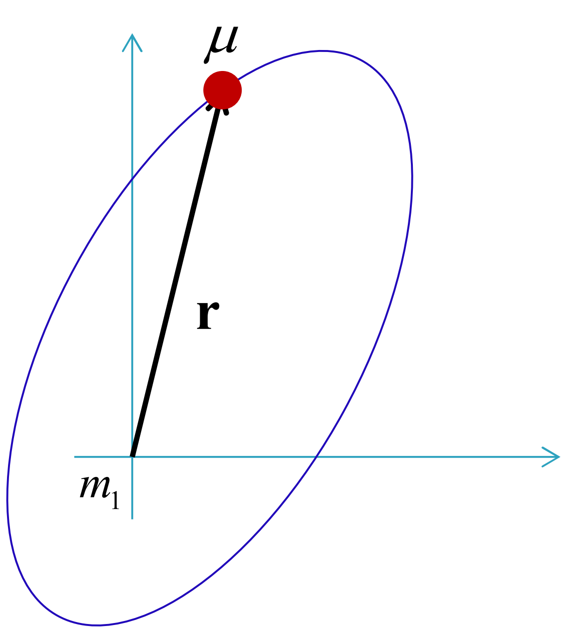
Then, by pick an inertial ref. frame (CM frame) in which CM is *not* moving $\dot{\mathbf{R}} = 0$

The problem can be reduced to an effective single particle problem with μ moving in $U(r)$. :

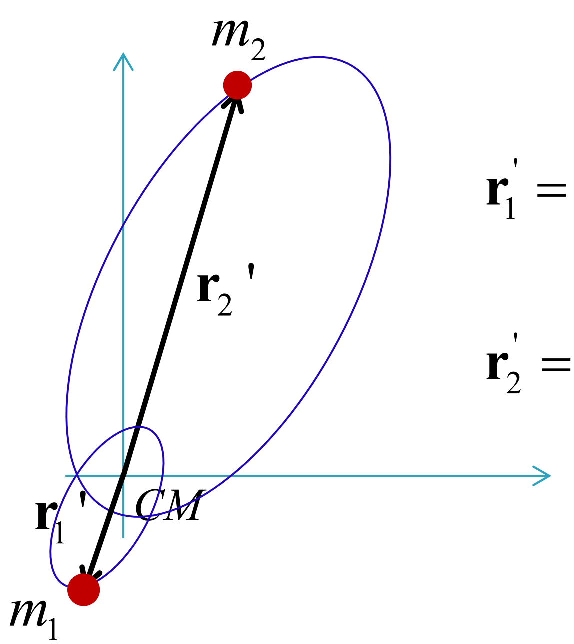
$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r), \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Reference Frames for Central Force Problems

Motion in the **relative coordinate frame** with $\left(\mathbf{R}\ \text{(fixed)}, \mathbf{r}\right)$ (3 dofs) :



Effective one body problem with reduced mass μ and relative position $\mathbf{r}$ (dist from m_1 to m_2).



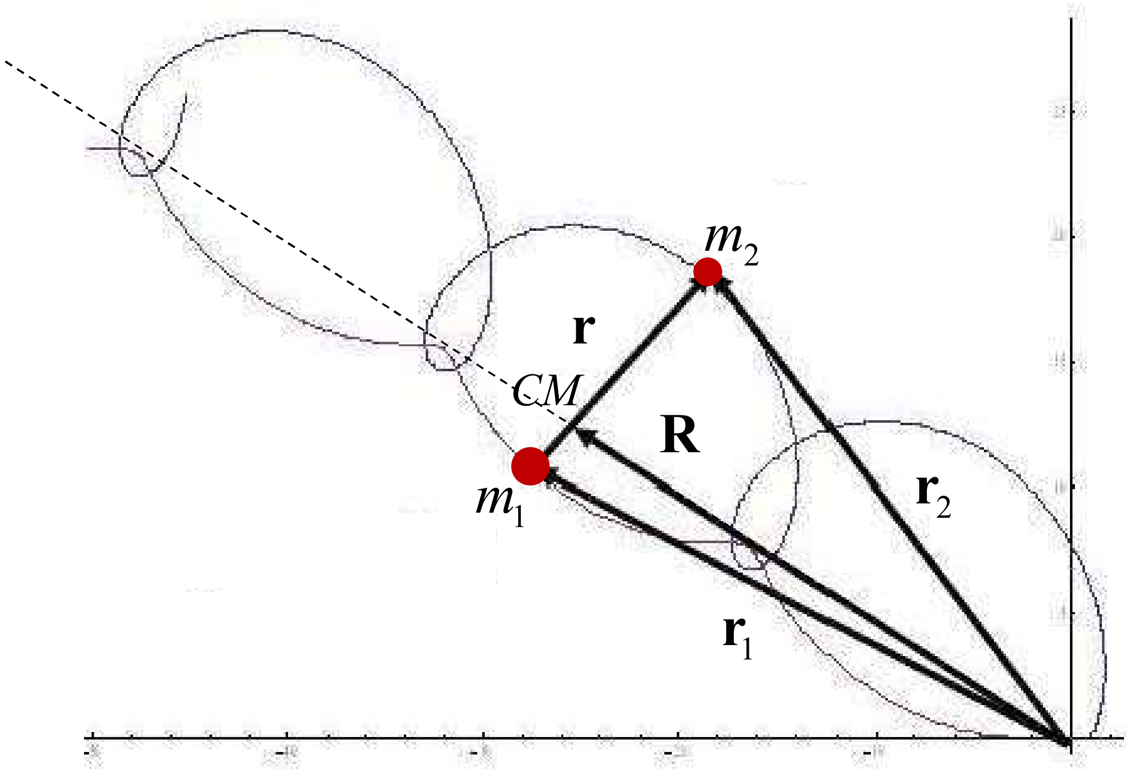
$$\mathbf{r}_1' = \frac{-m_2}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}$$

Motion of m_1 and m_2 in CM frame

Reference Frames for Central Force Problems

Motion in **original** $(\mathbf{r}_1, \mathbf{r}_2)$ **space** with 6 dofs (not all independent):



m_1 and m_2 circle each other in space as their CM move with constant velocity along a fixed direction.

Graphical Analysis of Central Force Problem

Using the concept of an **effective potential**, one can get a useful *qualitative* understanding of the problem without actually integrating!

Let consider the *r* equation:

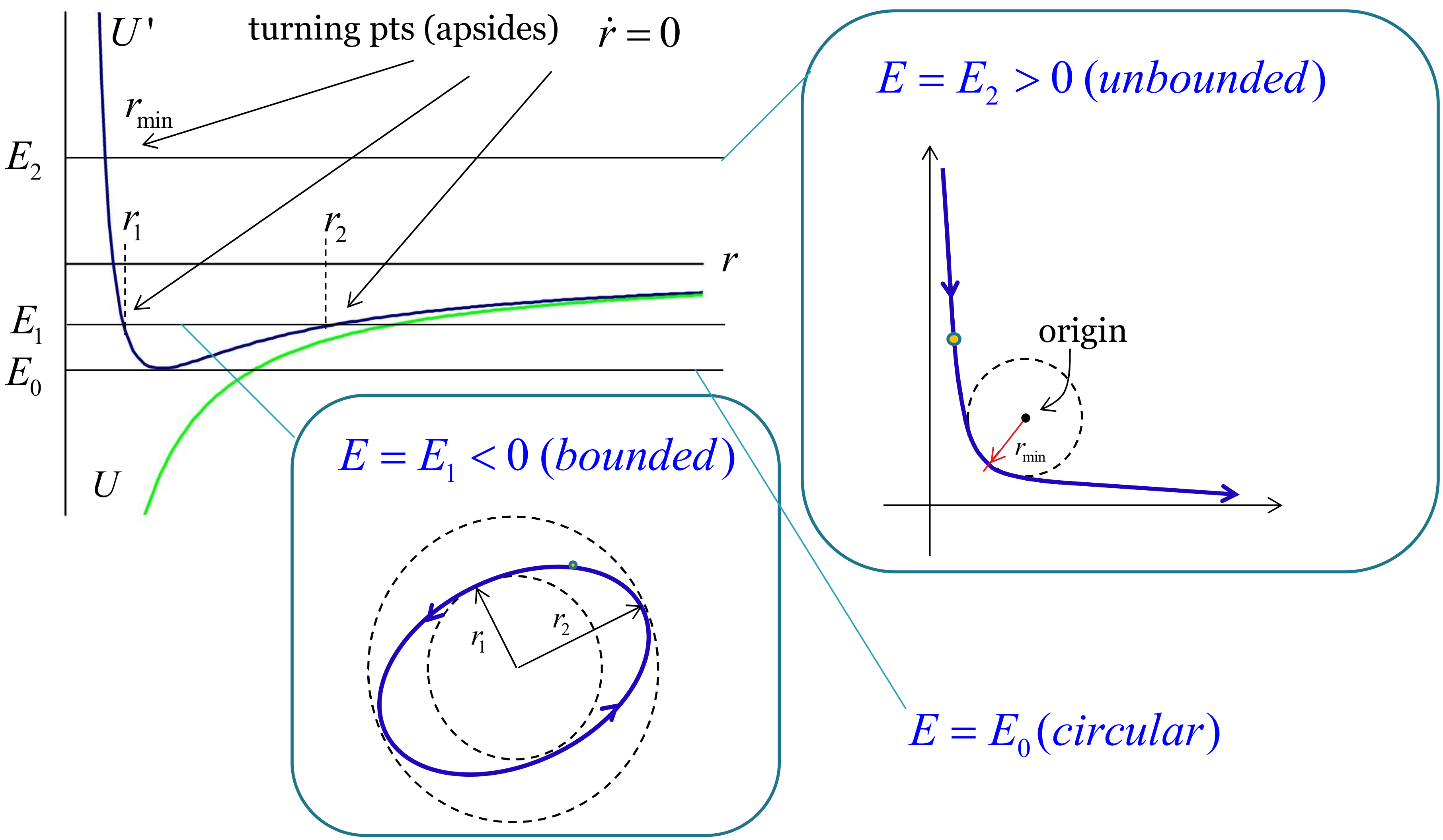
$$m\ddot{r} = \overbrace{-\frac{dU}{dr} + \frac{l^2}{mr^3}}$$

The last two terms combined can be considered as an effective force $f'(r)$

This looks like a 1D problem: a single particle moving in 1 dimension under the influence of an effective force and effective potential,

$$f'(r) = -\frac{dU}{dr} + \frac{l^2}{mr^3} \qquad U'(r) = U(r) + \frac{l^2}{2mr^2}$$

Central Force Problem: Inverse Square Force



Kepler Orbit Equation $r = r(\theta)$

The Kepler orbit equation in terms of the two constants of motion E and l:

$$r(\theta) = \frac{\alpha}{1+\varepsilon\cos(\theta-\theta')}$$

ε is the eccentricity

with $$\alpha = \frac{l^2}{mk}$$

$$\varepsilon = \sqrt{1+\frac{2El^2}{mk^2}}$$

Focus-Directrix Formulation: Summary

Summary on conic sections for the Kepler orbits:

$$\begin{array}{lll} \varepsilon > 1 & E > 0 & \textit{hyperbola} \\ \varepsilon = 1 & E = 0 & \textit{parabola} \\ 0 < \varepsilon < 1 & E < 0 & \textit{ellipse} \\ \varepsilon = 0 & E = -\dfrac{mk^2}{2l^2} & \textit{circle} \end{array} \left. \vphantom{\begin{array}{l} a \\ b \\ c \end{array}} \right\} E = -\frac{k}{2a}$$

$$r = \frac{\alpha}{1 + \varepsilon \cos\theta}$$

$$\alpha = \frac{l^2}{mk} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Formulation of the Problem: Quadratic Form of L

With V and T given near q_{0j}, we can now form the Lagrangian:

$$L = T - V = \frac{1}{2}\left(T_{jk}\,\dot{\eta}_j\dot{\eta}_k - V_{jk}\,\eta_j\eta_k\right) \qquad \text{(E's sum rule)}$$

- the deviation η_j from the equilibrium q_{0j} is our generalized coords
- T_{jk} and V_{jk} evaluated at q_{0j} are *constant* $n \times n$ square matrices
- dynamic near ANY equilibrium → **quadratic forms** in T and V
- coupling information among diff dofs are encoded in the cross terms
- GOAL for the following analysis is to

To find a coordinate transformation so that in the new coords

T_{jk} and V_{jk} diagonalize *simultaneously*

→ the problem *decoupled* !

Formulation of the Problem: Eigenvalue Equation

The *generalized* eigenvalue problem: $\mathbf{Va} = \lambda \mathbf{Ta}$

can be solved by solving the Characteristic equation:

$$\det\left(\mathbf{V} - \omega^2 \mathbf{T}\right) = 0$$

- $\lambda = \omega^2$ are the **eigenvalues** or **resonant frequencies** of the problem
- $\mathbf{a}$ is the **eigenvector** which will determine the relative relations among the generalized coords η_j in the different eigenmodes (**normal modes**)

V and **T** are *real* and *symmetric* ⇒

- $\lambda = \omega^2$ is real
- $\mathbf{a}$ is real and "orthogonal"

Euler's and Chasles's Theorems

Useful general principle for the analysis of rigid bodies…

Euler's Theorem:

⇨ A general displacement of a rigid body with 1 pt fixed in space is equivalent to a rotation about some axis.

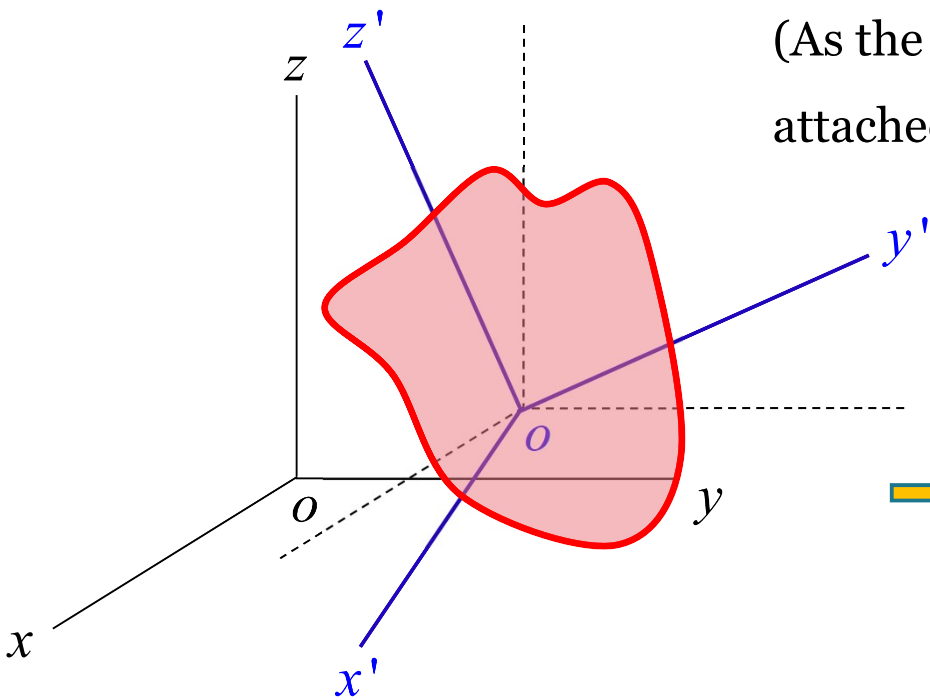
Chasles's Theorem:

⇨ A general displacement = a translation + a rotation

"Fixed" and "Body" Axes for a rigid body

We will use two sets of coordinates:

- 1 set of external "fixed" (lab) coordinates (x, y, z) (unprimed)
- 1 set of internal "body" coordinates (x', y', z') (primed)



(As the name implied, the "body" axes are attached to the rigid body.)

⇨ 3 coords to specify the origin *o* of the "body" axes wrt the fixed system

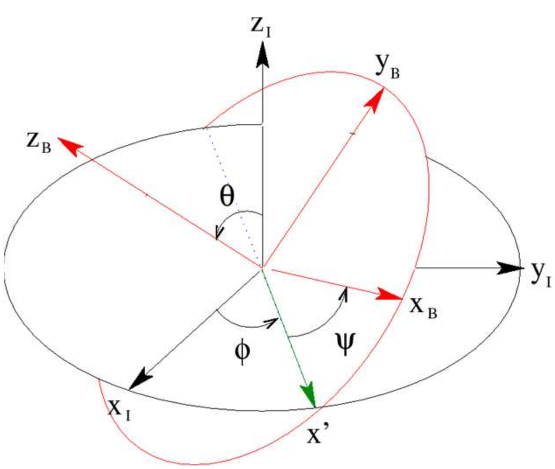
⇨ 3 coords to specify the *orientation* of the "body" axes wrt to the translated "fixed" system (dotted axes).

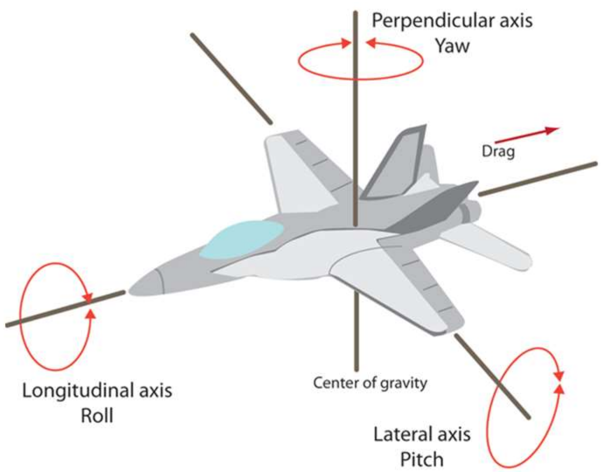
Orientation of the "Body" Axes

There are many ways to describe the *orientation* of the "body" axes…

➡ Most often used: Euler's angles (later) ≠ but similar to (roll, pitch, yaw)

- A sequence of 3 rotations in a standard order to get from the fixed axes (shifted) to the body axes.
- This gives a choice for the 3 orientation coordinates

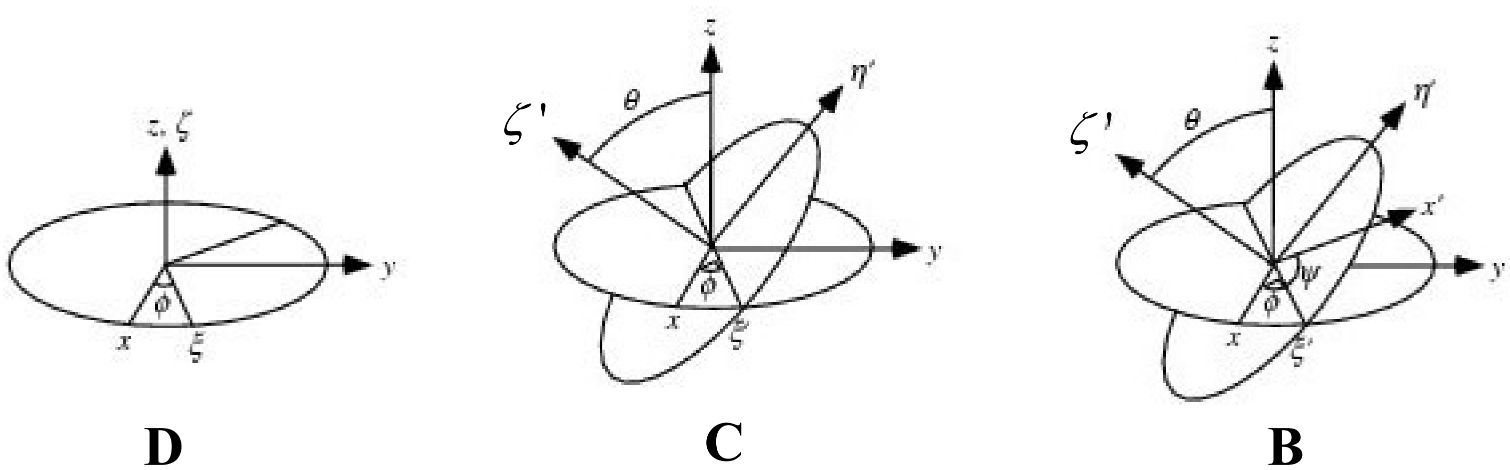




Euler's Angles

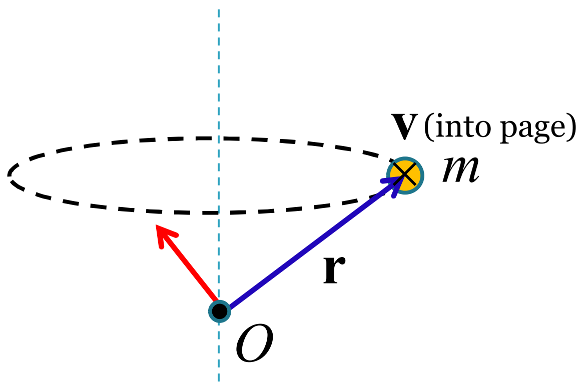
- There are many conventions… We will choose one called the

Euler's Angles ϕ, θ, ψ consisting of a particular *sequence* of 3 rotations (**D**, **C**, **B**) along three principle axes:



L and N depends on the Choice of Origin

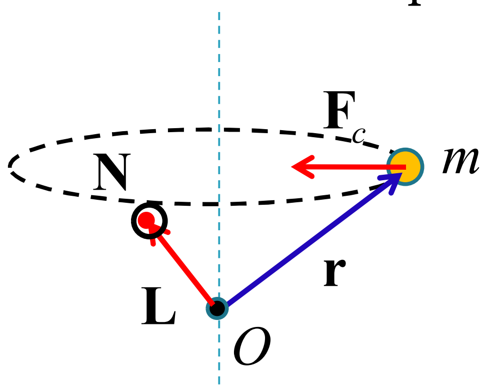
In general, **L** and **N** depends on the choice of the origin,



$\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ points as shown

Most importantly, it is no longer a constant, i.e, $\dfrac{d\mathbf{L}}{dt} = \mathbf{N} \neq 0$

And, there must be a torque acting on it.



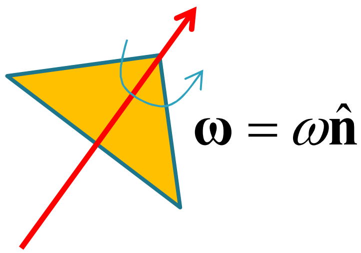
$\mathbf{N} = \mathbf{r} \times \mathbf{F}_c \neq 0$ (out of page)

(**L** changes in direction as given by **N**)

Non-Collinear Relation between L and ω

When a rigid object spins about one of its axes of symmetry, the dynamical equations are simple:

$$T_{rot} = \frac{1}{2} I \omega^2 \qquad \mathbf{L} = I\boldsymbol{\omega}$$



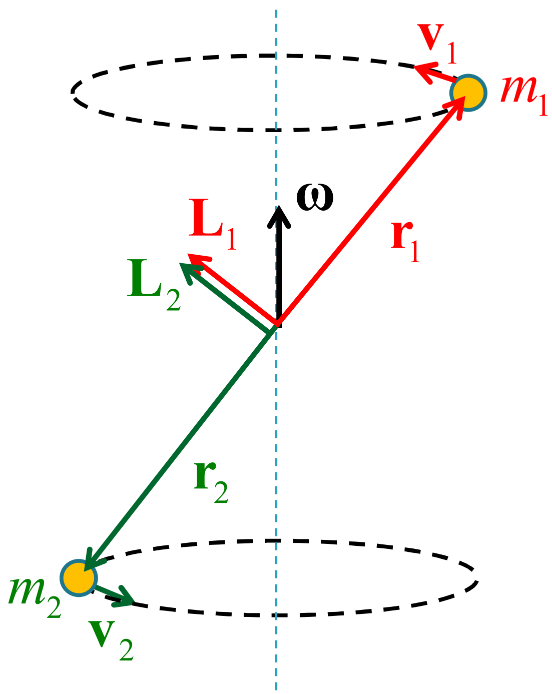
where - the moment of inertia I is simply a scalar

- **L** and ω point in the SAME direction

And, **L** and ω are not collinear unless when a rigid body is rotating about one of its **Principle Axes** of rotation.

Non-Collinear Relation between L and ω

And, **L** and ω are not collinear unless when a rigid body is rotating about one of its **Principle Axes** of rotation.



- Here, the total angular momentum is,

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$$

$$\mathbf{L} = m_1 \left(\mathbf{r}_1 \times \mathbf{v}_1 \right) + m_2 \left(\mathbf{r}_2 \times \mathbf{v}_2 \right)$$

- Notice that **L** and ω do NOT point in the same direction.

- In fact, **L** is changing in time so that there must be a torque:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

(In general, one just can't expect an object to rotate about a particular chosen axis in space w/o **N**.)

Also, the rotational KE doesn't work out to be quite $T_{rot} = \frac{1}{2} I \omega^2$ either.

T and L for a Rigid Body

Summary:

$$T_{rot} = \frac{1}{2}\omega_i I_{ij} \omega_j$$ (T_{rot} is a scalar and the sum is over both *i* and *j*)

$$L_i = I_{ij}\omega_j$$ (**L** is a vector and the sum is only over *j*)

$$I_{ij} = \sum_{\alpha} m_\alpha \left(\delta_{ij} x_k^\alpha x_k^\alpha - x_i^\alpha x_j^\alpha\right)$$ (I_{ij} is a tensor and the sum is only over α)

Notes:

- The moment of inertia **I** is no longer a scalar as in the simpler case: $\mathbf{L} = I\boldsymbol{\omega}$

- As we have seen in our earlier example, **L** and ω are not necessary collinear !

- However, we can still formally write,

$$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}$$

Euler's Equations

If we use the body axes to align with the Principal Axes of the rigid body, then one can description of the dynamics of the body *in its body frame* by the Euler's Equations:

$$
\begin{aligned}
I_1\dot{\omega}_1 - \left(I_2 - I_3\right)\omega_2\omega_3 &= N_1 \\
I_2\dot{\omega}_2 - \left(I_3 - I_1\right)\omega_3\omega_1 &= N_2 \\
I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2 &= N_3
\end{aligned}
$$

Symmetric Top in an Uniform Gravity Field

- Using an effective potential approach to analyze the resulting EOM,

$$E' = \frac{I_1}{2}\dot{\theta}^2 + V_{eff}(\theta) \qquad V_{eff}(\theta) = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

We can describe the dynamics in three distinct modes of motion:

- 3rd Euler angle: $\dot{\psi}$ = spin about the body's symmetry axis
- 1st Euler angle: $\dot{\phi}$ = precession of the body's symmetry axis about the space x_3' ($\hat{\mathbf{z}}$)axis

(have seen in torque free case)

- 2nd Euler angle: $\dot{\theta}$ = nutation (bobbing up & down) of the body symmetry axis (this is new)

- So, our effective 1D treatment will tell us about this new behavior (nutation) !