CHAPTER 2

LAGRANGIAN FORMULATION OF MECHANICS

CHAPTER OVERVIEW

Chapter 1 set the stage for the rest of the book: it reviewed Newton's equations and the basic concepts of Newton's formulation of mechanics. The discussion in that chapter was applied mostly to dynamical systems whose arena of motion is Euclidean three-dimensional space, in which it is natural to use Cartesian coordinates. However, we referred on occasion to other situations, such as one-dimensional systems in which a particle is not free to move in Euclidean 3-space but only in a restricted region of it. Such a system is said to be *constrained*: its arena of motion, or, as we shall define below, its *configuration manifold*, turns out in general to be neither Euclidean nor three dimensional (nor $3N$-dimensional, if there are N particles involved). In such cases the equations of motion must include information about the forces that give rise to the constraints.

In this chapter we show how the equations of motion can be rewritten in the appropriate configuration manifold in such a way that the constraints are taken into account from the outset. The result is the *Lagrangian formulation* of dynamics (the equations of motion are then called *Lagrange's equations*). We should emphasize that the physical content of Lagrange's equations is the same as that of Newton's. But in addition to being logically more appealing, Lagrange's formulation has several important advantages.

Perhaps the first evident advantage is that the Lagrangian formulation is easier to apply to dynamical systems other than the simplest. Moreover, it brings out the connection between conservation laws and important symmetry properties of dynamical systems. Of great significance is that Lagrange's equations can be derived from a variational principle, a method that turns out to be extremely general and applicable in many branches of physics. One of the reasons for studying classical mechanics is to understand the Lagrangian formulation, for many equations of physics are conventionally formulated in Lagrangian terms and many conservation laws are understood also in Lagrangian terms, through their connection with symmetries. Some of the topics we mention here will be put off until Chapter 3.

2.1 CONSTRAINTS AND CONFIGURATION MANIFOLDS

In this section we change from Cartesian coordinates to others, which are more useful for dealing with dynamical systems. The new coordinates are chosen in a way that depends on the particular dynamical system for which they will be used (but they are nevertheless called *generalized coordinates*); they are adapted to that system and are more or less natural coordinates for it. The properties of the system that determine the choice are geometric: they are the number of freedoms and the shape, or topology, of the region in which the system is free to move (e.g., whether it is a sphere or an inclined plane). This region is determined by the constraints placed upon the system; it is called the *configuration manifold* $\mathbb{Q}$. The new coordinates, called the q^α, will lie on $\mathbb{Q}$, and their number will be the number of freedoms, which is also the dimension of $\mathbb{Q}$. In this section we do two things: 1. explain the idea of the configuration manifold and 2. describe the change from the Cartesian coordinates to the q^α.

2.1.1 CONSTRAINTS

We start with an example. Think of a sphere rolling on a curved surface under the action of gravity. The sphere consists of many particles whose motion is correlated so that they always form a rigid sphere and so that there is always one of them in contact with the surface and, as the body is rolling, instantaneously at rest. The forces on the sphere are far from simple. They are composed of the forces internal to the sphere (which keep it rigid), the forces applied to it by the surface on which it is rolling (which keep it in contact with that surface and prevent it from sliding), and the force of gravity. The force of gravity is known a priori, but the others, the constraining forces, are not. What is known is that under the action of gravity and the forces of constraint the body remains on the surface and continues to roll. It might seem that to describe the motion completely one would have to find the constraining forces, but it will be shown that the opposite is true, that the motion can be obtained from the gravitational force and from knowing the geometric constraints (i.e., of the shape of the surface and of the fact of rigidity); the forces of constraint, if needed, are easier to find later. This seemingly simple example of a sphere rolling on a curved surface is actually quite complicated. Most of the time we will be dealing with much simpler constraints. We now proceed to generalize this example.

CONSTRAINT EQUATIONS

The motion of a dynamical system is often constrained by external agents applying forces that are initially unknown. What is known is the geometric effect of such agents, or rather their effect combined with those applied forces that are known. Suppose one is dealing with a system of N particles and that the constraints are given by a set of K *constraint equations* of the form

$$f_I(\mathbf{x}_1, \ldots, \mathbf{x}_N, t) = 0, \quad I = 1, \ldots, K < 3N, \tag{2.1}$$

where the $\mathbf{x}_i$ are the position vectors of the N particles. The f_I are assumed to be differentiable functions of their arguments, and the t dependence describes the known way in which the constraints vary with time, independent of the motion of the particles (for instance, in the example of the rolling sphere with which we started this chapter, the surface on which it is rolling could be waving).

Constraints given by equations like (2.1) are called *holonomic* (meaning essentially integrable, from the Greek). More general constraints depend also on the velocities (rolling constraints are among them); they are given by equations of the form

$$f_I(\mathbf{x}_1, \ldots, \mathbf{x}_N; \dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_N, t) = 0, \quad I = 1, \ldots, K < 3N. \tag{2.2}$$

There exist constraints that appear to be velocity dependent but are actually differential equations that can be integrated to give simply holonomic constraints. When this is not the case, velocity-dependent constraints are *nonholonomic*. In any case, it should be clear that holonomic constraints are a special case of this more general type. Finally, there are other types of constraints entirely that are not even given by equations, for example, those given by expressions of the form

$$f_I(\mathbf{x}_1, \ldots, \mathbf{x}_N, t) < 0, \quad I = 1, \ldots, K < 3N, \tag{2.3}$$

as in the case of particles restricted to a certain region of space. An example is a particle constrained to remain within a container of some given shape. Although this kind of constraint will be mentioned on occasion in the book, it will not be treated in any generality. In this chapter we will deal only with holonomic constraints.

CONSTRAINTS AND WORK

How one deals with constraints can be illustrated (Fig. 2.1) by the relatively simple example of a point particle in 3-space restricted to a surface whose equation is

$$f(\mathbf{x}, t) = 0. \tag{2.4}$$

In this example $N = 1$ (a single particle) and $K = 1$ (a single constraint equation). The Newtonian equation of motion of the particle is

$$m\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{C}, \tag{2.5}$$

where $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$ is the known external force and $\mathbf{C}$ is the unknown force of constraint that the surface exerts on the particle. We now have four equations, namely (2.4) and the three components of (2.5), for six unknown functions of the time, the three components of $\mathbf{x}$ and the three components of $\mathbf{C}$. Clearly this is not enough to determine the motion. The problem arises from the physical fact that there are many possible constraint forces $\mathbf{C}$ that will keep the particle on the surface [i.e., will lead to motions that satisfy (2.4)]. For example, suppose that the surface is a stationary plane and that a constraint force $\mathbf{C}$ h

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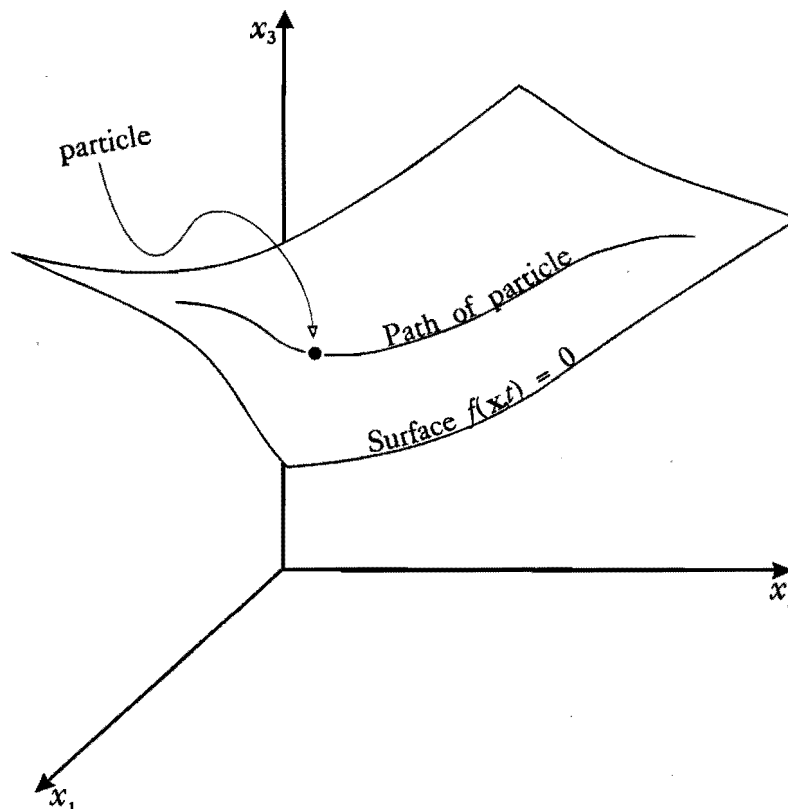constraint forces C
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FIGURE 2.1

A particle in 3-space constrained to a two-dimensional surface given by an equation of the form $f(\mathbf{x}, t) = 0$.

been found that will keep the particle on the plane. Now add to C another force that is parallel to the plane (friction is an example of such a force) and call the sum C′. It is clear then that C′ will also keep the particle on the plane but accelerating at some different rate, for the only difference between the two is a force along the plane. For a curved surface the argument is similar: to a constraining force C can be added a force that at each point is parallel to the surface, and the resulting force will still constrain the particle to the surface. What is needed here is some physical input that will allow us to choose among the different possibilities for C.

This input will be obtained by what seems at first an arbitrary choice (as will be seen later, what this does physically is to place restrictions on the work done by the constraint forces): the forces parallel to the surface will be eliminated by choosing C to be perpendicular (normal) to the surface. The way to obtain a vector perpendicular to a surface is the following:

Let $f(\mathbf{x}, t) = $ const. be the equation of any surface; then $\nabla f(\mathbf{x}, t)$ is a vector perpendicular to the surface at position $\mathbf{x}$ and time t, provided that $\nabla f \neq 0$ on the surface. If $\nabla f \equiv 0$ on the surface, of course, the procedure we are outlining here will not work. To avoid this difficulty, we shall require that the constraint has been written in such a

$$\nabla f \neq 0 \quad \text{on the } f = 0 \text{ surface.} \tag{2.6}$$

For example, the two equations $f_a(\mathbf{x}) = \mathbf{s} \cdot \mathbf{x} = 0$ and $f_b(\mathbf{x}) = (\mathbf{s} \cdot \mathbf{x})^2 = 0$ constrain a particle to the same plane, but on that plane $\nabla f_a = \mathbf{s}$, while $\nabla f_b = 0$, so only $f_a = ($ is acceptable. (If $K > 1$ and $N > 1$, the requirement is that the matrix of the $\partial f_I / \partial x^\alpha$ be at least of rank K; see the book's appendix for the definition of rank. The x^α are the $3N$ components of N position vectors of the particles.)

The constraint force perpendicular or normal to the surface (often called a *normal force* and written $\mathbf{N}$ in place of $\mathbf{C}$, but we will stick with $\mathbf{C}$) can therefore be written

$$\mathbf{C} = \lambda \nabla f(\mathbf{x}, t), \tag{2.7}$$

where λ can be any number, in particular a function of t. This removes the mathematical difficulty, because now the four equations involve only four unknown functions, namely $\lambda(t)$ and the three components of $\mathbf{x}(t)$. But the physical implications of this assumption have yet to be understood, so we now turn aside in order to understand them.

Assume that the external force depends on a potential: $\mathbf{F} = -\nabla V(\mathbf{x}, t)$. Then expressing $\mathbf{C}$ through (2.7) and taking the dot product with $\dot{\mathbf{x}}$ on both sides of (2.5) leads to

$$m\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \equiv \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = -\nabla V \cdot \dot{\mathbf{x}} + \lambda \nabla f \cdot \dot{\mathbf{x}}. \tag{2.8}$$

Now suppose that $\mathbf{x}(t)$ is a solution of the equations of motion. Then since the particle remains on the surface, $f(\mathbf{x}(t), t) = 0$, and therefore $df/dt = 0$. But

$$\frac{df}{dt} = \nabla f \cdot \dot{\mathbf{x}} + \frac{\partial f}{\partial t},$$

and similarly

$$\frac{dV}{dt} = \nabla V \cdot \dot{\mathbf{x}} + \frac{\partial V}{\partial t}.$$

From these equations and (2.8) it follows that

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V\right] = \frac{\partial V}{\partial t} - \lambda\frac{\partial f}{\partial t}. \tag{2.9}$$

This means that the total energy E of the particle changes if V or f are explicit functions of the time (i.e., if the potential depends on the time or if the constraint surface is moving). We will not deal at this point with time-dependent potential energy functions, and therefore if (but not only if) the surface moves, the total energy changes.

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The relation between movement of the surface and energy change can be understood physically. If the energy is changing, that is, if $dE/dt \neq 0$, the work–energy theorem [Eq. (1.39)] implies that there is work being done on the system; as we are assuming that $\partial V/\partial t = 0$, Eq. (2.9) implies that the work is performed by the surface. To see how the surface does this work, suppose first that it is not moving. Since $\mathbf{C}$ is normal to the surface, it is always perpendicular to the velocity $\dot{\mathbf{x}}$, and thus $\mathbf{C} \cdot \dot{\mathbf{x}} = 0$: the rate at which work is done by the constraint force is zero. If the surface moves, however, the particle velocity need not be tangent to the surface, as shown in Fig. 2.2, and even if $\mathbf{C}$ is perpendicular to the surface $\mathbf{C} \cdot \dot{\mathbf{x}} \neq 0$: the surface through $\mathbf{C}$ can do work at a non-zero rate.
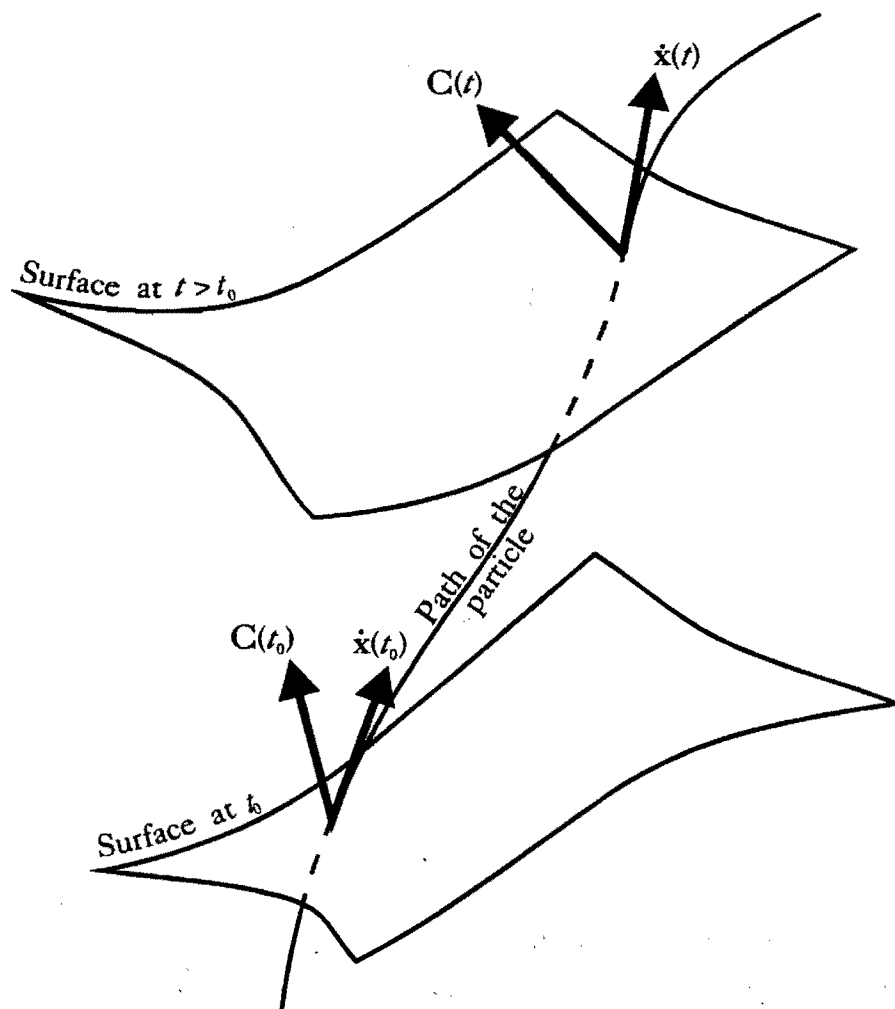

FIGURE 2.2

A two-dimensional constraint surface that depends on time. Although the constraint force $\mathbf{C}$ is always normal to the surface, the angle between $\mathbf{C}$ and the particle's velocity vector $\dot{\mathbf{x}}$ is not a right angle. Therefore the constraint force can do work on the particle.

The physical content of Eq. (2.7), i.e., the assumption that **C** is normal to the surface should now be clear: it is that the forces of constraint do no work. On the other hand, in reality, most surfaces exert forces which have tangential components such as friction and therefore do work. Thus for the time being we are excluding frictional forces. But there can be other constraint forces, nondissipative ones, that have components parallel to the surface of constraint yet do no work (they need only be perpendicular to the velocity, like magnetic forces on charged particles). We are excluding those also. When those conditions are satisfied, the surface is called *smooth*.

2.1.2 GENERALIZED COORDINATES

We now return to the equations of motion for a single particle constrained to a surface:

$$m\ddot{\mathbf{x}} = \mathbf{F} + \lambda \nabla f, \tag{2.10}$$

$$f(\mathbf{x}, t) = 0. \tag{2.11}$$

These are solved by first eliminating $\lambda(t)$. Since $\lambda \nabla f$ is perpendicular to the surface, one can eliminate λ by taking only those components of (2.10) that are tangent to it. For this purpose let τ be an *arbitrary* vector tangent to the surface at $\mathbf{x}$ at time t, that is, a vector restricted only by the condition that $\tau \cdot \nabla f = 0$ (recall that $\nabla f \neq 0$ on the constraint surface). Then the dot product of (2.10) with τ yields

$$(m\ddot{\mathbf{x}} - \mathbf{F}) \cdot \tau = 0. \tag{2.12}$$

This equation says only that $m\ddot{\mathbf{x}} - \mathbf{F}$ is perpendicular to the surface at $\mathbf{x}$ at time t. Such a tangent vector τ is now found at each point $\mathbf{x}$ of the surface and at all times t (that is, τ is a vector function of $\mathbf{x}$ and of t); thus we obtain a τ-dependent equation for $m\ddot{\mathbf{x}} - \mathbf{F}$. Since τ is an arbitrary vector tangent to the surface, there are two linearly independent vectors at each point $\mathbf{x}$ and hence two linearly independent vector functions of $\mathbf{x}$ and t. Therefore this procedure yields not one but two equations for $m\ddot{\mathbf{x}} - \mathbf{F}$. But three equations are needed if one wants to find the vector function $\mathbf{x}(t)$. The third equation that is available is Eq. (2.11). The result is essentially a set of second-order differential equations for the components of $\mathbf{x}(t)$. If one wants to know the force $\mathbf{C}$ of constraint, one can solve for $\mathbf{x}(t)$ and return to (2.5).

So far we have found how to write the equations of motion for a single particle with a single holonomic constraint. We now generalize to a system of N particles with K independent holonomic constraints. Because we will often be using double indices without summing and because triple indices will sometimes occur, we drop the summation convention for a while. The analog of Eq. (2.5), the equation of motion of the ith particle, is (no sum on i)

$$m_i \ddot{\mathbf{x}}_i = \mathbf{F}_i + \mathbf{C}_i, \tag{2.13}$$

and the constraints are given by (2.1). As before, the constraints fail to determine the $\mathbf{C}_i$ completely, and we add the assumption of smoothness by writing the analog

of (2.7), namely

$$\mathbf{C}_i = \sum_{I=1}^{K} \lambda_I \nabla_i f_I, \tag{2.14}$$

where ∇_i is the gradient with respect to the position vector $\mathbf{x}_i$ of the ith particle and the $\lambda_I(t)$ are K functions, which are as yet unknown. Like λ in the one-particle case, the λ_I will be eliminated in solving the problem. We leave to the reader (see Problem 1) the task of proving that, in analogy with the one-particle case, if the potential V satisfies $\partial V / \partial t = 0$ the total change in energy is given by

$$\frac{dE}{dt} = -\sum_I \lambda_I \frac{\partial f_I}{\partial t}, \tag{2.15}$$

so that the forces of constraint do work only if the constraint functions depend on t.

Now let the $\boldsymbol{\tau}_i$ be N arbitrary vectors "tangent to the surface," that is, vectors restricted only by the condition that

$$\sum_{i=1}^{N} \boldsymbol{\tau}_i \cdot \nabla_i f_I = 0, \quad I = 1, \ldots, K. \tag{2.16}$$

If, as required in the discussion around Eq. (2.6), the matrix of the $\partial f_I / \partial x^\alpha$ is of rank K, this equation gives K independent relations among the $3N$ components of the N vectors $\boldsymbol{\tau}_i$, so that only $3N - K$ of the components are independent. Then the dot product of (2.13) with $\boldsymbol{\tau}_i$, summed over i, yields [use (2.14) and (2.16)]

$$\sum_i (m_i \ddot{\mathbf{x}}_i - \mathbf{F}_i) \cdot \boldsymbol{\tau}_i = 0. \tag{2.17}$$

This equation, the analog of (2.12), is sometimes called D'Alembert's principle. Through it, the $3N - K$ independent components of the $\boldsymbol{\tau}_i$ lead to $3N - K$ independent relations. Equations (2.1) provide K other relations, so that there are $3N$ in all from which the $3N$ components of the $\mathbf{x}_i$ can be obtained.

The problem now is to find a suitable algorithm for picking vectors $\boldsymbol{\tau}_i$ that satisfy (2.16). We will do this by sharpening the analogy to the one-particle case. In the one-particle case $\boldsymbol{\tau}$ was an arbitrary vector tangent to the surface of constraint. In the N-particle case there is no surface of constraint, so the $\boldsymbol{\tau}_i$ are not readily visualized. But Eq. (2.1) defines a $(3N - K)$-dimensional hypersurface in the $3N$-dimensional Euclidean space $\mathbb{E}^{3N}$ of the components of the $\mathbf{x}_i$, and the dynamical system is constrained to this hypersurface. That is, as the system moves and the $\mathbf{x}_i$ keep changing, the point in $3N$-space described by the collection of all the components of the $\mathbf{x}_i$ remains always on this hypersurface. We could therefore call it the *configuration hypersurface* of the dynamical system, but for reasons that will be explained in Section 2.4, we will call it its *configuration manifold* $\mathbb{Q}$. Start with N tangent vectors $\boldsymbol{\tau}_i$ that satisfy (2.16). Their $3N$ components define a ($3N$-component)

vector in $\mathbb{E}^{3N}$ that is a kind of generalized tangent vector to the configuration manifold, for in Eq. (2.16) the sum is not only over i, as indicated by the summation sign, but also over the three components of each $\boldsymbol{\tau}_i$, as indicated by the dot product. Thus just as $\boldsymbol{\tau} \cdot \nabla f = 0$ defines a 3-vector tangent to the $f = 0$ surface, Eq. (2.16) defines a $3N$-vector tangent to the $f_I = 0$ hypersurface $\mathbb{Q}$ (see Problem 2).

In these terms picking the $\boldsymbol{\tau}_i$ to satisfy (2.16) means picking the generalized tangent vector in $3N$ dimensions. This vector will be found in several steps. The first will be to define what are called *generalized coordinates* q^α in the $3N$-space (superscripts rather than subscripts are generally used for these coordinates) for which $\mathbb{Q}$ is a coordinate hypersurface. Consider a *region* of $\mathbb{E}^{3N}$ that contains a point $\mathbf{x}_i$ of $\mathbb{Q}$, and let q^α, $\alpha = 1, \ldots, 3N$, be new coordinates in that region, a set of invertible functions of the $\mathbf{x}_i$:

$$\begin{aligned} q^\alpha &= q^\alpha(\mathbf{x}_1, \ldots, \mathbf{x}_N, t), \\ \mathbf{x}_i &= \mathbf{x}_i(q^1, \ldots, q^{3N}, t) \end{aligned} \tag{2.18}$$

for $\mathbf{x}_i$ in that region. Equations (2.18) define a *transformation* between the $\mathbf{x}_i$ and the q^α. Invertibility means that the Jacobian of the transformation is nonsingular. (The Jacobian of the transformation is the matrix whose elements are the $\partial q^\alpha / \partial x^\beta$, where the x^β are the $3N$ components of the N vectors $\mathbf{x}_i$.)

Assume further that the q^α are continuous and, because accelerations will lead to second derivatives, twice continuously differentiable functions. The first object will be to pick the q^α so that the equations of constraint become trivial (i.e., reduce to the statement that some of the q^α are constant). Then if the equations of motion are written in terms of the q^α (invertibility guarantees that they can be), those q^α that are constant will drop out. This is done by choosing the q^α so that K of them (we choose the last K) depend on the $\mathbf{x}_i$ through the functions appearing in the constraint equations. Suppressing any t dependence, we write

$$q^{n+I}(\mathbf{x}) = R_I(f_1(\mathbf{x}), \ldots, f_K(\mathbf{x})), \quad I = 1, \ldots, K, \tag{2.19}$$

where $\mathbf{x}$ stands for the collection of the $\mathbf{x}_i$ and $n = 3N - K$ is the dimension of the configuration manifold $\mathbb{Q}$; n is also the number of freedoms. Equations (2.19) are the last K of the $3N$ equations that give the q^α in terms of the $\mathbf{x}_i$, and as such they too must be invertible, which means that it must be possible to solve them for the $f_I = f_I(q^{n+I}, \ldots, q^{n+K})$. When the constraint conditions are imposed, they force the last K of the q^α to be constants independent of the time:

$$q^{n+I} = R(0, \ldots, 0). \tag{2.20}$$

This is what we mean by the constraint equations becoming trivial in these coordinates. Since the last K of the q^α remain fixed as the motion proceeds, the problem reduces to finding how the rest of the q^α, the first n, depend on the time.

The full set of q^α can be used as well as the $\mathbf{x}_i$ to define a point in $\mathbb{E}^{3N}$. That is what is meant by invertibility. Equation (2.20) restricts the point to lie in $\mathbb{Q}$: it makes the same

statement as does (2.18) but in terms of different coordinates. As the first n of the q^α vary in time, the point described by the full set moves about in $\mathbb{E}^{3N}$, but it remains on the configuration manifold, and therefore the first n of the q^α form a coordinate system on $\mathbb{Q}$. The first n of the q^α are called *generalized coordinates* of the dynamical system. Hence when the equations of motion are written in terms of the generalized coordinates, they will describe the way the system moves within the configuration manifold.

From now on, Greek indices run from 1 to $n = 3N - K$, rather than from 1 to $3N$.

2.1.3 EXAMPLES OF CONFIGURATION MANIFOLDS

In this subsection we give examples of configuration manifolds and generalized co-ordinates for some particular dynamical systems. In these examples and in most of what follows, Greek indices will run from 1 to n.

THE FINITE LINE

The finite line, which may be curved, applies to the motion of a bead along a wire of length l (Fig. 2.3a). In this case $N = 1$ and $K = 2$ (see Problem 2), so that $n = 1$ and α takes on only the single value 1 and may be dropped altogether. Here $\mathbb{Q}$ is of dimension 1 (the dimension being essentially the number of coordinates, the number of values that α takes on), and the coordinate system on it may be chosen so that the values of q range from $-l/2$ to $l/2$.

THE CIRCLE

The circle applies to the motion of a plane pendulum (Fig. 2.3b). Denote the circle by $\mathbb{S}^1$. Again $\mathbb{Q}$ is one dimensional, and the single generalized coordinate is usually taken to be the angle and is called θ rather than q. Typically the coordinates on the circle are chosen so that θ varies from $-\pi$ to π, or from 0 to 2π. But note that both of these choices have a problem: in each of them there is one point with two coordinate values. In the first choice the point with coordinate π is the same as the one with coordinate $-\pi$, and in the second, this is true for the coordinates 0 and 2π. This lack of a unique relationship between the points of $\mathbb{Q}$ and a coordinate system is an important property of manifolds and will be treated in some detail later (see Section 2.4).

THE PLANE

The plane applies to the motion of a particle on a table (Fig. 2.3c). As before, $N = 1$, but now $K = 1$, so that $n = 2$. The coordinates are conveniently chosen to be the usual plane Cartesian, plane polar, or other familiar coordinates.

THE TWO-SPHERE $\mathbb{S}^2$

The surface of the sphere applies to the motion of a spherical pendulum, which consists of a point mass attached to a weightless rigid rod that is free to rotate about a fixed point in a uniform gravitational field (Fig. 2.3d). The coordinates usually chosen on $\mathbb{S}^2$ are the azimuth angle ϕ (corresponding to the longitude on the globe of the Earth), which varies