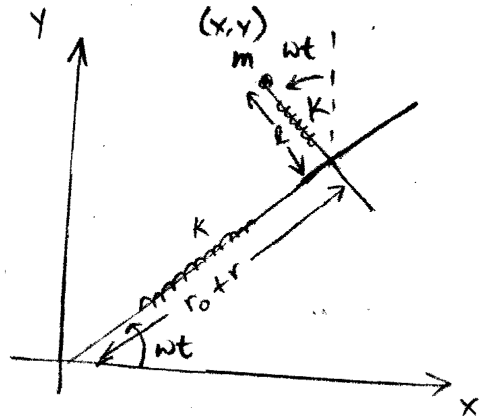
2.21



We will consider the system in the following two generalized coordinate systems:

- "lab" system: standard Cartesian (x, y)
- "rotating" system: (r, ℓ) as indicated in the graph shown.

These two coordinate systems are related by the following point transformation:

$$\begin{cases} x = (r_0+r)\cos\omega t - \ell \sin\omega t \\ y = (r_0+r)\sin\omega t + \ell\cos\omega t \end{cases} \qquad (*)$$

We will also need the inverse transformation:

$$x\cos\omega t = (r_0+r)\cos^2\omega t - \ell\sin\omega t\cos\omega t$$

$$\oplus \quad y\sin\omega t = (r_0+r)\sin^2\omega t + \ell\sin\omega t\cos\omega t$$

$$r = x\cos\omega t + y\sin\omega t - r_0$$

$$-x\sin\omega t = -(r_0+r)\cos\omega t\sin\omega t + \ell\sin^2\omega t$$

$$\oplus \quad y\cos\omega t = (r_0+r)\cos\omega t\sin\omega t + \ell\cos^2\omega t$$

$$\ell = -x\sin\omega t + y\cos\omega t$$

Now, we are ready to calculate the various quantities in these two coordinates:

a) The total energy of the system is given by,

$$E = T + V$$

we can calculate this in the "lab" coordinates:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$V = \frac{1}{2} k(r^2 + \ell^2)$$

$$= \frac{1}{2} k\left(x^2\cos^2\omega t + y^2\sin^2\omega t + r_0^2 + \cancel{2xy\cos\omega t\sin\omega t} - 2xr_0\cos\omega t - 2yr_0\sin\omega t + x^2\sin^2\omega t + y^2\cos^2\omega t - \cancel{2xy\sin\omega t\cos\omega t}\right)$$

$$= \frac{1}{2} k\left(x^2 + y^2 + r_0^2 - 2r_0(x\cos\omega t + y\sin\omega t)\right)$$

$$E = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k\left(x^2 + y^2 + r_0^2 - 2r_0(x\cos\omega t + y\sin\omega t)\right)$$

Since V doesn't depend on $\dot{x}, \dot{y}$, $h = E$.

Since $\frac{\partial L}{\partial t} \neq 0$, $h = E$ is not conserved.

b). The Jacobi integral (energy function) in the "lab" coordinates is:

$$h(x,y) = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2+y^2+r_0^2 - 2r_0(x\cos\omega t + y\sin\omega t))$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$h(x,y) = m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}k(x^2+y^2+r_0^2 - 2r_0(x\cos\omega t + y\sin\omega t))$$

$$h(x,y) = E(x,y) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}k(x^2+y^2+r_0^2 - 2r_0(x\cos\omega t + y\sin\omega t))$$

Since L depends on time explicitly,

$\longrightarrow$ $h(x,y) = E(x,y)$ is not a conserved quantity!

c). In the "rotating" coordinates:

From the point transformations on p.1 ⊛, we have

$$\begin{cases} \dot{x} = -(r_0+r)\omega\sin\omega t + \dot{r}\cos\omega t - \dot{\ell}\sin\omega t - \ell\omega\cos\omega t \\ \dot{y} = (r_0+r)\omega\cos\omega t + \dot{r}\sin\omega t + \dot{\ell}\cos\omega t - \ell\omega\sin\omega t \end{cases}$$

$$\dot{x}^2+\dot{y}^2 = (r_0+r)^2\omega^2\sin^2\omega t + \dot{r}^2\cos^2\omega t + \dot{\ell}^2\sin^2\omega t + \ell^2\omega^2\cos^2\omega t$$
$$+(r_0+r)^2\omega^2\cos^2\omega t + \dot{r}^2\sin^2\omega t + \dot{\ell}^2\cos^2\omega t + \ell^2\omega^2\sin^2\omega t$$

$$+2\big[-\cancel{(r_0+r)\dot{r}\omega\sin\omega t}\cos\omega t + (r_0+r)\dot{\ell}\omega\sin^2\omega t + \cancel{(r_0+r)\ell\omega^2\sin\omega t}\cos\omega t$$
$$-\dot{r}\cancel{\dot{\ell}\sin\omega t\cos\omega t} - \dot{r}\ell\omega\cos^2\omega t + \cancel{\dot{\ell}\ell\omega\sin\omega t\cos\omega t}\big]$$
$$+2\big[\cancel{(r_0+r)\dot{r}\omega}\sin\omega t\cos\omega t + (r_0+r)\dot{\ell}\omega\cos^2\omega t - \cancel{(r_0+r)\ell\omega^2}\sin\omega t\cos\omega t$$
$$+\dot{r}\cancel{\dot{\ell}\sin\omega t}\cos\omega t - \dot{r}\ell\omega\sin^2\omega t - \cancel{\dot{\ell}\ell\omega\sin\omega t\cos\omega t}\big]$$

$$= (r_0+r)^2\omega^2 + \dot{r}^2 + \dot{\ell}^2 + \ell^2\omega^2 + 2(r_0+r)\dot{\ell}\omega - 2\dot{r}\ell\omega$$

$$(\dot{r}-\ell\omega)^2$$

$$((r_0+r)\omega+\dot{\ell})^2$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\left(((r_0+r)\omega+\dot{\ell})^2 + (\dot{r}-\ell\omega)^2\right)$$

$$V = \frac{1}{2}k(r^2+\ell^2)$$ ← simple in the rotating coordinates.

$$L = T-V = \frac{1}{2}m\left(((r_0+r)\omega+\dot{\ell})^2 + (\dot{r}-\ell\omega)^2\right) - \frac{1}{2}k(r^2+\ell^2)$$

This is the Lagrangian in the rotating coordinates

The Jacobi Integral is:

$$h(r,\ell) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\ell}\frac{\partial L}{\partial \dot{\ell}} - L$$

$$\frac{\partial L}{\partial \dot{r}} = m(\dot{r} - \ell \omega) \qquad \frac{\partial L}{\partial \dot{\ell}} = m((r_0+r)\omega + \dot{\ell})$$

$$h(r,\ell) = m\dot{r}(\dot{r} - \ell\omega) + m\dot{\ell}((r_0+r)\omega + \dot{\ell}) - \frac{1}{2}m((r_0+r)\omega + \dot{\ell})^2$$
$$- \frac{1}{2}m(\dot{r} - \ell\omega)^2 + \frac{1}{2}k(r^2+\ell^2)$$

Consider the 1st & 4th terms :

$$m\dot{r}(\dot{r} - \ell\omega) - \frac{1}{2}m(\dot{r} - \ell\omega)^2$$
$$= (\dot{r} - \ell\omega)\left(m\dot{r} - \frac{1}{2}m\dot{r} + \frac{1}{2}m\ell\omega\right)$$
$$= (\dot{r} - \ell\omega)\left(\frac{1}{2}m\dot{r} + \frac{1}{2}m\ell\omega\right)$$
$$= \frac{1}{2}m(\dot{r} - \ell\omega)(\dot{r} + \ell\omega) = \frac{1}{2}m(\dot{r}^2 - \ell^2\omega^2)$$

Similarly, 2nd & 3rd terms can be regrouped,

$$m\dot{\ell}((r_0+r)\omega + \dot{\ell}) - \frac{1}{2}m((r_0+r)\omega + \dot{\ell})^2$$
$$= ((r_0+r)\omega + \dot{\ell})\left(m\dot{\ell} - \frac{1}{2}m(r_0+r)\omega - \frac{1}{2}m\dot{\ell}\right)$$
$$= ((r_0+r)\omega + \dot{\ell})\left(\frac{1}{2}m\right)(\dot{\ell} - (r_0+r)\omega)$$
$$= \frac{1}{2}m(\dot{\ell}^2 - (r_0+r)^2\omega^2)$$

Putting all these together, we have

$$h(r,\ell) = \frac{1}{2}m(\dot{r}^2 + \dot{\ell}^2) + \frac{1}{2}k(r^2+\ell^2) - \frac{1}{2}m((r_0+r)^2 + \ell^2)\omega^2$$

Since L does not depend on time explicitly in (r, ℓ),

- $h(r,\ell)$ is a conserved quantity!
- But, since the point transformation ⊛ depends on time explicitly, $h(r,\ell) \neq E$, i.e. $h(r,\ell)$ is not the total energy.

Lastly, also notice that the last quadratic term in $h(r,\ell)$ is in the form of $\frac{1}{2} I \omega^2$

where $I = m(r_0+r)^2 + m\ell^2$

So that $h(r,\ell)$ has a clear physical interpretation in this rotating frame:

The motion is basically are two "decoupled" harmonic motion with a kinetic energy term $\left(\frac{1}{2}m(\dot{r}^2+\dot{\ell}^2)\right)$ & a potential energy term associated with the springs $\left(\frac{1}{2}k(r^2+\ell^2)\right)$ minus the rotational KE due to the constant rotation $\left(\frac{1}{2}m((r_0+r)^2+\ell^2)\omega^2\right)$.

2.24 :

$$L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$$ a 1D harmonic oscillator (1)

Suppose we are guessing the soln for the equation of motion as

$$x(t) = \sum_j a_j \cos(j\omega t)$$ a Fourier Series for $x(t)$

where ω is the natural frequency of the motion to be determined.

Note: Similar to our derivation of the Euler-Lagrange equation, $x(t)$ are our "test functions" with $\{a_j\}$ as the parameters in our variations.

To apply the Hamilton Principle, we first write the action for this problem:

$$I(\{a_j\}) = \int_0^T L(x,\dot{x};t)\,dt$$ $T = \frac{2\pi}{\omega}$ is one period of the oscillation

Then, we are interested in finding the stationary solution $x(t)$ such that

$$\delta I = \frac{\partial I}{\partial a_j} da_j = 0$$ for all $j = 0, \dots, \infty$

Note: different from our previous problems in lecture where we have one parameter α, we have infinite # of parameters.

let consider a particular a_j, we have

$$\frac{\partial I}{\partial a_j} da_j = \int_0^T \left(\frac{\partial L}{\partial x} \frac{\partial x}{\partial a_j} + \frac{\partial L}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial a_j} \right) da_j \, dt \qquad (2)$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$x(t) = \sum_\ell a_\ell \cos(\ell \omega t)$$

$$\dot{x}(t) = -\sum_\ell a_\ell (\ell \omega) \sin(\ell \omega t)$$

$$\frac{\partial x}{\partial a_j} = \cos(j\omega t) \qquad \frac{\partial \dot{x}}{\partial a_j} = -(j\omega)\sin(j\omega t)$$

Putting these together, we have:

$$\frac{\partial I}{\partial a_j} da_j = \int_0^T \Big[-k\Big(\sum_\ell a_\ell \cos(\ell\omega t) \cos(j\omega t)\Big) + m\Big(\sum_\ell a_\ell (\ell\omega) \sin(\ell\omega t)\, (j\omega) \sin(j\omega t)\Big)\Big] da_j \, dt$$

$$= \sum_\ell \Big[-k a_\ell \int_0^T \cos(\ell\omega t)\cos(j\omega t)\, dt + m a_\ell (\ell j \omega^2) \int_0^T \sin(\ell\omega t)\sin(j\omega t)\, dt \Big] da_j$$

Using the orthogonality properties of the sines & cosines:

$$\int_0^T \sin(\ell\omega t)\sin(j\omega t)\, dt = \int_0^T \cos(\ell\omega t)\cos(j\omega t)\, dt = \begin{cases} T/2, & \ell = j \\ 0, & \ell \neq j \end{cases},$$

We will pick out only the $\ell = j$ term in the sum.

$$\frac{\partial I}{\partial a_j} da_j = a_j(-k + mj^2w^2)\, da_j$$

Setting this to zero, we have the condition for all j:

$$a_j(-k + mj^2w^2) = 0$$

For a nonvanishing solution, we need at least one of the a_j to be nonzero!

And, a_{j_0} will be nonzero only if the j_0th factor on the right is zero, i.e.,

$$-k + mj_0^2 w^2 = 0$$

And, this gives $w^2 = \frac{k}{mj_0^2}$

→ Note that if we pick $w^2 = \frac{k}{mj_0^2}$ for a particular j_0, then all other a_j, $j \neq j_0$ must be zero!

Now, look back at $x(t) = \sum_j a_j \cos(jwt)$,

we have $x(t) = a_{j_0} \cos\left(j_0 \frac{1}{j_0}\sqrt{\frac{k}{m}}\, t\right)$

$= a_{j_0} \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

where a_{j_0} will be determined by initial cond.

This means that it doesn't matter what actual value for j_0 we choose, we will get the same form of solution for

$$x(t) = a \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

For simplicity, we can simply pick $j_0 = 1$.

so that we will have a non zero solution

$$x(t) = a \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

only if $a_j = 0$ for all $j \neq 1$

and $\omega^2 = \frac{k}{m}$.