Problem 1 : particle with constant mass m:

$$T = \sum_i \frac{1}{2} m v_i^2$$

$$\frac{dT}{dt} = \sum_i \frac{1}{2} m \frac{d}{dt}(v_i^2)$$

$$= \sum_i \frac{1}{2} m \left(v_i \frac{dv_i}{dt} + \frac{dv_i}{dt} v_i \right)$$

$$= \sum_i \left(m \frac{dv_i}{dt} \right) v_i$$

$$= \sum_i \frac{d}{dt}(m v_i) v_i = \vec{F} \cdot \vec{v}$$

If mass is a function of time:

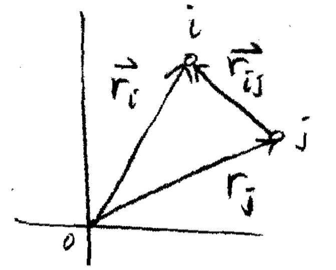
$$\frac{d(mT)}{dt} = \sum_i \frac{1}{2} \frac{d}{dt} (m^2 v_i^2)$$

$$= \sum_i \frac{1}{2} \frac{d}{dt} (p_i^2)$$

$$= \sum_i \frac{1}{\cancel{2}} \left(\cancel{2} p_i \frac{dp_i}{dt} \right)$$

$$= \vec{F} \cdot \vec{p}$$

1.2



$$M^2R^2 = (M\vec{R})\cdot(M\vec{R})$$

$$= \left(\sum_i m_i \vec{r}_i\right)\cdot\left(\sum_j m_j \vec{r}_j\right) = \sum_i\sum_j m_i m_j (\vec{r}_i\cdot\vec{r}_j) \qquad \circledast$$

From the graph above, we have

$$\vec{r}_j - \vec{r}_i = \vec{r}_{ij}$$

$$r_{ij}^2 = |\vec{r}_{ij}|^2 = (\vec{r}_j - \vec{r}_i)\cdot(\vec{r}_j - \vec{r}_i)$$

$$= r_j^2 - 2\vec{r}_i\cdot\vec{r}_j + r_i^2$$

$$\Rightarrow \quad \vec{r}_i\cdot\vec{r}_j = \frac{1}{2}\left(r_i^2 + r_j^2 - r_{ij}^2\right)$$

plug this into $\circledast$, we have

$$M^2R^2 = \frac{1}{2}\sum_i\sum_j m_i m_j \left(r_i^2 + r_j^2 - r_{ij}^2\right)$$

$$= \frac{1}{2}\sum_i\sum_j m_i m_j r_i^2 + \frac{1}{2}\sum_i\sum_j m_i m_j r_j^2 - \frac{1}{2}\sum_i\sum_j m_i m_j r_{ij}^2$$

$$= \frac{1}{2}\left(\sum_j m_j\right)\left(\sum_i m_i r_i^2\right) + \frac{1}{2}\left(\sum_i m_i\right)\left(\sum_j m_j r_j^2\right) - \frac{1}{2}\sum_i\sum_j m_i m_j r_{ij}^2$$

$$= \underline{M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2} \qquad (r_{ii} = 0)$$

1.3

external force on #1

particle 1 : $m_1 \ddot{\vec{r}}_1 = \vec{F}_1^{e} + \vec{F}_{21}$

particle 2 : $m_2 \ddot{\vec{r}}_2 = \vec{F}_2^{e} + \vec{F}_{12}$

internal forces between #1 & #2

external force on #2

$$(+) \Rightarrow \quad m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = \vec{F}_1^{e} + \vec{F}_2^{e} + \vec{F}_{21} + \vec{F}_{12}$$

$$\underline{M \vec{R}} = \vec{F}^{e} + \vec{F}_{21} + \vec{F}_{12}$$

Eq. 1.22 for the center of Mass.

So, in order for Eq. 1.22 to be correct, we need to have $\underline{\vec{F}_{21} = -\vec{F}_{12}}$.

this is the <u>weak form of Newton's 3rd law</u>.

Now, we take the cross products of the individual Equations of motion above:

$$\begin{cases} \vec{r}_1 \times (m \ddot{\vec{r}}_1) = \vec{r}_1 \times \dot{\vec{p}}_1 = \vec{r}_1 \times \vec{F}_1^{e} + \vec{r}_1 \times \vec{F}_{21} \\ \vec{r}_2 \times (m \ddot{\vec{r}}_2) = \vec{r}_2 \times \dot{\vec{p}}_2 = \vec{r}_2 \times \vec{F}_2^{e} + \vec{r}_2 \times \vec{F}_{12} \end{cases}$$

$$(+) \Rightarrow \quad \vec{r}_1 \times \dot{\vec{p}}_1 + \vec{r}_2 \times \dot{\vec{p}}_2 = \left(\vec{r}_1 \times \vec{F}_1^{e} + \vec{r}_2 \times \vec{F}_2^{e}\right) + \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12}$$

$$\dot{\vec{L}} = \vec{N}^{e} + \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12}$$

Again, in order to have Eq. 1.26 for the total angular momentum to be true, we need to have

$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = 0$$

From previous consideration on the total force alone, we have $\vec{F}_{21} = -\vec{F}_{12}$, this reduce the above condition to:

$$(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{12} = 0$$

$$\vec{r}_{12} \times \vec{F}_{21} = -\vec{r}_{12} \times \vec{F}_{12} = 0$$

This can only be true if

$$\vec{F}_{12} \;\&\; \vec{F}_{21} \parallel \vec{r}_{12} \;!$$

This is the strong form of Newton's 3rd law.

1.4 Recall:

If $\sum_{i}^{n} g_i(x_1, \dots, x_n)\, dx_i = 0$ is an exact different, the there exicts a function $G(x_1, \dots, x_n)$ such that $g_i = \dfrac{\partial G}{\partial x_i}$ ①

and we will have the following conditions:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial G}{\partial x_j \partial x_i} = \frac{\partial G}{\partial x_i \partial x_j} = \frac{\partial g_j}{\partial x_i}.$$

Now, for a constraint equation

$$\sum_{i}^{n} g_i(x_1, \dots, x_n)\, dx_i = 0$$ to be holonomic,

it has to be an exact differential or there exist an integrating factor $f(x_1, \dots, x_n)$

Such that $\sum_{i}^{n} (f g_i) dx_i$ will be an exact diff.,

i.e. $$\frac{\partial (f g_i)}{\partial x_j} = \frac{\partial (f g_j)}{\partial x_i}$$

For the constraints given in Eq. 1.39:

$$dx - a \sin\theta \, d\phi = 0$$

$$dy + a \cos\theta \, d\phi = 0$$

let assume $f(x, y, \theta, \phi)$ is the required integrating factor the first equation :

let consider the 1st constraint equation and the following notation :

$$g_x dx + g_y dy + g_\theta d\theta + g_\phi d\phi = dx - a\sin\theta d\phi = 0$$

then, we need to have

$$\frac{\partial}{\partial\theta}(f g_\phi) = \frac{\partial}{\partial\phi}(f g_\theta)$$

$$-a\frac{\partial}{\partial\theta}(f\sin\theta) = \underline{-af\cos\theta = 0}$$

There does not exist a non-zero $f(x, y, \theta, \phi)$ such that this can be true for all θ!

Similar argument applies to the second constraint and there does not exist an integrating function which can make it into an exact differential.