(11)

5.9 Please refer to the picture of the heavy top on P. 209 or the picture on next page. (1)

We are asked the derive the two constants of motion for the heavy top given by Eqs. 5.53 & 5.54 from the Euler's Equations.

$$5.53 \rightarrow \quad I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{const}$$

$$5.54 \rightarrow \quad (I_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = \text{const}$$

The Euler's Equation is given by:

$$(*) \begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \end{cases}$$

- With the body axes (x_1, x_2, x_3) chosen as shown along the principle axes of the heavy top, we have:

$$I_1 = I_2 \neq I_3$$

- By RHR, the torque $\vec{N}$ can be shown to be along the line of nodes and

$$\begin{cases} N_1 = N\cos\psi \\ N_2 = -N\sin\psi \\ N_3 = 0 \end{cases}$$

2

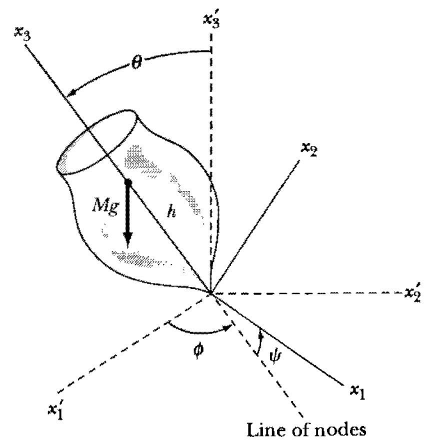


Figure: 11-15

Size: 13p9 x 14p6

Wadsworth
Thornton/Marion
Classical Dynamics 5/e
7973

Artist/Date: eh 3/13/03
Revs: BJ 4/03/03

<u>Note</u>: one can also get the components of the force (by gravity) along body axes by multiplying the full Euler angle transformation matrix to $\begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$.

$$\vec{F}_g = \underset{B}{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{C}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}} \underset{D}{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -mg\sin\theta \\ -mg\cos\theta \end{pmatrix} = \begin{pmatrix} -mg\sin\theta\sin\psi \\ -mg\sin\theta\cos\psi \\ -mg\cos\theta \end{pmatrix}$$

Then, $\vec{N} = \vec{r} \times \vec{F}_g$

$$= \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & r \\ F_g^1 & F_g^2 & F_g^3 \end{pmatrix} = \begin{pmatrix} -rF_g^2 \\ rF_g^1 \\ 0 \end{pmatrix}$$

$$\vec{N} = \begin{pmatrix} mgr\sin\theta\cos\psi \\ -mgr\sin\theta\sin\psi \\ 0 \end{pmatrix} = \begin{pmatrix} N\cos\psi \\ -N\sin\psi \\ 0 \end{pmatrix}$$

where, $N = rmg\sin\theta$

Now, we can simplify the Euler's Equation $\circledast$:

$$
\begin{cases}
I_1 \dot{\omega}_1 - (I_1 - I_3)\,\omega_2 \omega_3 = N \cos\psi \\
I_1 \dot{\omega}_2 - (I_3 - I_1)\,\omega_3 \omega_1 = -N \sin\psi \\
I_3 \dot{\omega}_3 = 0
\end{cases}
$$

- We can now use 4.87 to express the angular velocity $\vec{\omega}$ in the body axes in terms of the Euler's angle :

$$
\begin{cases}
\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\
\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\
\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}
\end{cases}
$$

- Substituting ω_3 into the 3rd equation, we immediately get

$$I_3 \frac{d}{dt} \left(\dot{\phi} \cos\phi + \dot{\psi} \right) = 0$$

$$\frac{d}{dt} \left(I_3 \left(\dot{\phi} \cos\phi + \dot{\psi} \right) \right) = 0$$

$$\Rightarrow \quad \underline{I_3 \left(\dot{\phi} \cos\phi + \dot{\psi} \right) = \text{const}} \quad \leftarrow$$

one of the constant of motion

Eq. 5.53.

Now, substitute ω_1 & ω_2 into the first two equations:

(1)
$$I_1 \frac{d}{dt}(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi) - (I_1 - I_3)(\dot\phi \sin\theta\cos\psi - \dot\theta \sin\psi)\cdot(\dot\phi\cos\theta + \dot\psi) = N\cos\psi$$

(2)
$$I_1 \frac{d}{dt}(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi) - (I_3 - I_1)(\dot\phi\cos\theta + \dot\psi)\cdot(\dot\phi \sin\theta\sin\psi + \dot\theta\cos\psi) = -N\sin\psi$$

☆ The goal is to combine these two equations in such a way that the RHS = 0. the LHS will be a <u>full</u> time derivative and the Then the expression inside the full time derivative will be a constant of motion.

(1) $\sin\psi$ $\oplus$ (2) $\cos\psi$:

$$I_1\left[\sin\psi \frac{d}{dt}(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi) + \cos\psi\frac{d}{dt}(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\right]$$
$$-(I_1 - I_3)\left[\begin{pmatrix}\sin\psi\,\dot\phi\sin\theta\cos\psi \\ -\dot\theta\sin^2\psi\end{pmatrix}\cdot(\dot\phi\cos\theta + \dot\psi) - \cos\psi(\dot\phi\cos\theta + \dot\psi)\begin{pmatrix}\dot\phi\sin\theta\sin\psi \\ +\dot\theta\cos^2\psi\end{pmatrix}\right] = 0$$

$\downarrow$ the $(I_1 - I_3)$ term

$$-(I_1 - I_3)\left[\cancel{(\dot\phi\sin\theta\sin\psi\cos\psi)\cdot(\dot\phi\cos\theta + \dot\psi) - (\dot\phi\sin\theta\sin\psi\cos\psi)\cdot(\dot\phi\cos\theta + \dot\psi)} - \dot\theta(\dot\phi\cos\theta + \dot\psi)(\sin^2\psi + \cos^2\psi)\right]$$

$$= (I_1 - I_3)\dot\theta(\dot\phi\cos\theta + \dot\psi)$$

The I_1 (first) term:

$$I_1\left[\sin\psi\left[\cancel{(\dot\phi\sin\theta)\cos\psi\,\dot\psi} + \frac{d}{dt}(\dot\phi\sin\theta)\sin\psi + \frac{d}{dt}(\dot\theta\cos\psi)\right]\right.$$
$$\left.+\cos\psi\left[\cancel{-(\dot\phi\sin\theta)\sin\psi\,\dot\psi} + \frac{d}{dt}(\dot\phi\sin\theta)\cos\psi - \frac{d}{dt}(\dot\theta\sin\psi)\right]\right]$$

$$= I_1\left[\frac{d}{dt}(\dot\phi\sin\theta)(\sin^2\psi+\cos^2\psi) + \sin\psi(-\dot\theta\sin\psi\,\dot\psi + \cancel{\ddot\theta\cos\psi})\right.$$
$$\left.-\cos\psi(\dot\theta\cos\psi\,\dot\psi + \cancel{\ddot\theta\sin\psi})\right]$$

$$= I_1\left[\frac{d}{dt}(\dot\phi\sin\theta) - \dot\theta\dot\psi\right]$$

- Putting these two terms back together, we have

$$I_1\left[\frac{d}{dt}(\dot\phi\sin\theta) - \dot\theta\dot\psi\right] + (I_1 - I_3)\,\dot\theta(\dot\phi\cos\theta + \dot\psi) = 0$$

$$I_1\frac{d}{dt}(\dot\phi\sin\theta) \cancel{-I_1\dot\theta\dot\psi} + I_1\dot\theta\dot\phi\cos\theta \cancel{+I_1\dot\theta\dot\psi} - I_3\dot\theta\dot\phi\cos\theta - I_3\dot\theta\dot\psi$$

$$I_1\frac{d}{dt}(\dot\phi\sin\theta) + I_1\dot\theta\dot\phi\cos\theta - I_3\dot\theta(\dot\phi\cos\theta + \dot\psi) = 0$$

- multiply $\sin\theta$ on both sides:

$$I_1\sin\theta\frac{d}{dt}(\dot\phi\sin\theta) + I_1\dot\phi\underbrace{\dot\theta\cos\theta}_{\frac{d}{dt}(\sin\theta)}\sin\theta - I_3(\dot\theta\sin\theta)(\dot\phi\cos\theta + \dot\psi) = 0$$

$$\downarrow$$

$$I_1\frac{d}{dt}(\sin\theta(\dot\phi\sin\theta)) \qquad + I_3\frac{d}{dt}(\cos\theta)(\dot\phi\cos\theta + \dot\psi)$$

– We already know that $I_3(\dot\phi\cos\theta + \dot\psi) = \text{const}$,

so that we can absorb it into the $\frac{d}{dt}(\cos\theta)$ term :

So, we have

$$I_1 \frac{d}{dt}\left(\sin\theta\,(\dot\phi\sin\theta)\right) + I_3\frac{d}{dt}\left(\cos\theta\,(\dot\phi\cos\theta + \dot\psi)\right) = 0$$

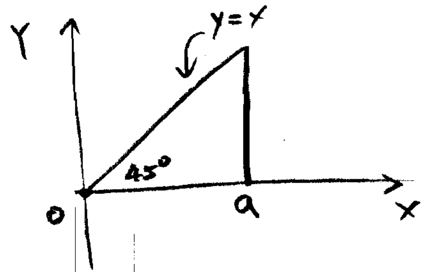
$$\Rightarrow \quad \frac{d}{dt}\left[I_1\dot\phi\sin^2\theta + I_3\dot\phi\cos^2\theta + I_3\cos\theta\,\dot\psi\right] = 0$$

Finally, we have

$$\underbrace{\left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot\phi + I_3\dot\psi\cos\theta = \text{const}}$$

which is the desired 2nd constant of motion.

5.15 For simplicity in integration, we will pick the following coordinate system for the rigid flat triangle.



← In this coordinate system, 2D integrals can be written simply as:

$$\int_0^a dx \int_0^x dy$$

- We will then use the parallel axis theorem to find the moments of inertia at the center of mass.

- Let the uniform mass density be σ per area, then the total mass M of the triangle is:

$$\underline{M} = \sigma \int_0^a dx \int_0^x dy = \sigma \int_0^a x\, dx = \underline{\tfrac{1}{2}\sigma a^2}$$

- Now, we will find the CM:

$$M X_{cm} = \sigma \int_0^a \int_0^x x\, dx\, dy = \sigma \int_0^a x^2\, dx$$

$$= \tfrac{1}{3}\sigma a^3 = \underline{\tfrac{2}{3} a M}$$

$$M Y_{cm} = \sigma \int_0^a \int_0^x y\, dx\, dy = \sigma \int_0^a \frac{x^2}{2}\, dx$$

$$= \tfrac{1}{6}\sigma a^3 = \underline{\tfrac{1}{3} a M}$$

Now, we calculate the moments of inertia with respect to the this coordinate system with the axis of rotation at the tip of the triangle (the origin). ②

$$I_{xx} = \sigma \int_0^a \int_0^x y^2 \, dx\, dy = \sigma \int_0^a \frac{x^3}{3} dx = \frac{1}{12} \sigma a^4 = \frac{1}{6} M a^2$$

$$I_{yy} = \sigma \int_0^a \int_0^x x^2 \, dx\, dy = \sigma \int_0^a x^3 \, dx = \frac{1}{4} \sigma a^4 = \frac{1}{2} M a^2$$

$$I_{xy} = -\sigma \int_0^a \int_0^x x y \, dx\, dy = -\sigma \int_0^a x \frac{x^2}{2} dx = -\frac{1}{8} \sigma a^4 = -\frac{1}{4} M a^2$$

$$I_{zz} = \sigma \int_0^a \int_0^x (x^2+y^2) \, dx\, dy = \sigma \int_0^a \left(x^2 \cdot x + \frac{x^3}{2}\right) dx$$

$$= \frac{1}{\cancel{4}} \sigma \frac{\cancel{4}}{3} a^4 = \frac{2}{3} M a^2$$

$$I_{xz} = \sigma \int_0^a \int_0^x x z_0 \, dx\, dy = \sigma \int_0^a \int_0^x y z_0 \, dx\, dy = 0$$

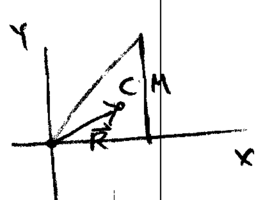
triangle lays flat at $z_0 = 0$.

$$\Rightarrow \quad \underset{\approx}{I} = \begin{pmatrix} 1/6 & -1/4 & 0 \\ -1/4 & 1/2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} M a^2$$

with axis of rotation at origin.

- Now we use parallel axis theorem to find $\underset{\approx}{I}_{cm}$ at CM:

$$I_{ij}^{CM} = I_{ij} - M(\delta_{ij} R^2 - R_i R_j)$$



$$I_{xx}^{cm} = \frac{1}{6}Ma^2 - M\left((\cancel{X_{cm}^2} + Y_{cm}^2) - \cancel{X_{cm}^2}\right)$$

$$= \frac{1}{6}Ma^2 - M\left(\frac{1}{9}a^2\right) = \left(\frac{3-2}{18}\right)Ma^2 = \frac{1}{18}Ma^2$$

$$I_{yy}^{cm} = \frac{1}{2}Ma^2 - M\left((X_{cm}^2 + \cancel{Y_{cm}^2}) - \cancel{Y_{cm}^2}\right)$$

$$= \frac{1}{2}Ma^2 - \frac{4}{9}Ma^2 = \frac{9-8}{18}Ma^2 = \frac{1}{18}Ma^2$$

$$I_{xy}^{cm} = \frac{-1}{4}Ma^2 + M\left(X_{cm}Y_{cm}\right)$$

$$= -\frac{1}{4}Ma^2 + \frac{2}{9}Ma^2 = \left(\frac{-9+8}{36}\right)Ma^2 = -\frac{1}{36}Ma^2$$

$$I_{zz}^{cm} = \frac{2}{3}Ma^2 - M\left(X_{cm}^2 + Y_{cm}^2\right)$$

$$= \frac{2}{3}Ma^2 - M\left(\frac{1}{9}a^2 + \frac{4}{9}a^2\right) = \left(\frac{6-5}{9}\right)Ma^2 = \frac{1}{9}Ma^2$$

$$\Rightarrow \quad \underline{\underline{I}}^{cm} = \begin{pmatrix} \frac{1}{18} & -\frac{1}{36} & 0 \\ -\frac{1}{36} & \frac{1}{18} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix} Ma^2$$

To get to the principle moments of inertia and principle axes, we need to find the eigenvalues & eigenvectors of $\underline{\underline{I}}^{cm}$.

$\underline{\lambda}$:

$$\det\left(\underline{\underline{I}}^{cm} - \lambda \underline{\underline{1}}\right) = 0$$

$$\left(\frac{1}{9} - \lambda\right)\left(\left(\frac{1}{18} - \lambda\right)^2 - \left(\frac{1}{36}\right)^2\right) = 0$$

④

$$\lambda = \frac{1}{9} \quad \& \quad \frac{1}{18} - \lambda = \pm\frac{1}{36}$$

$$\lambda = \frac{1}{18} \pm \frac{1}{36}$$

$$= \frac{3}{36} \text{ or } \frac{1}{36}$$

$$\lambda = \frac{1}{12} \text{ or } \frac{1}{36}$$

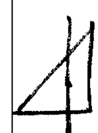
For $\lambda = \frac{1}{9}$,

$$\begin{pmatrix} -\frac{1}{18} & -\frac{1}{36} & 0 \\ -\frac{1}{36} & -\frac{1}{18} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\Rightarrow \quad 2x + y = 0$ and $x + 2y = 0$

$\Rightarrow \quad x = y = 0$

$\Rightarrow$ eigenvector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



(out of page)

$\rightarrow$ So, we have $I_3 = \frac{1}{9} Ma^2$ & the principle axis is $\hat{z}$.

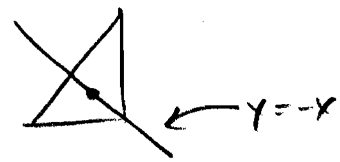
For $\lambda = \frac{1}{12}$, $\quad I_1 = \frac{1}{12} Ma^2$

$$\begin{pmatrix} \frac{1}{18} - \frac{1}{12} & -\frac{1}{36} & 0 \\ -\frac{1}{36} & \frac{1}{18} - \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{9} - \frac{1}{18} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \quad -\frac{1}{36}x - \frac{1}{36}y = 0 \quad \Rightarrow \quad x + y = 0 \quad \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$z = 0$$

→ So, we have $I_1 = \frac{1}{12} M a^2$ and the principle axis is



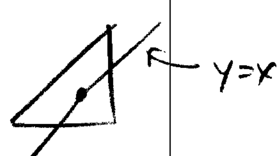
For $\lambda = \frac{1}{36}$, $I_2 = \frac{1}{36} M a^2$

$$\begin{pmatrix} \frac{1}{18} - \frac{1}{36} & -\frac{1}{36} & 0 \\ -\frac{1}{36} & \frac{1}{18} - \frac{1}{36} & 0 \\ 0 & 0 & \frac{1}{9} - \frac{1}{36} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{36} x - \frac{1}{36} y = 0 \quad \Rightarrow \quad x - y = 0 \quad \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

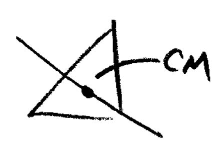
$$z = 0$$

→ So, we have $I_2 = \frac{1}{36} M a^2$, and



Summary :

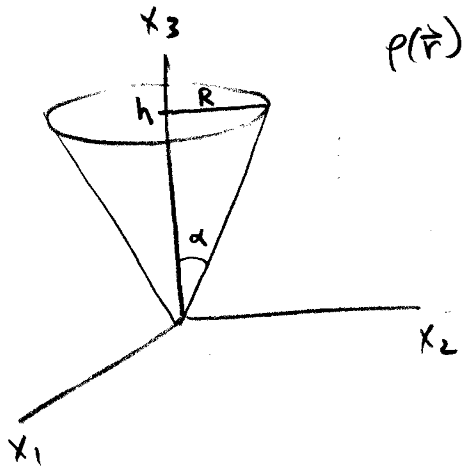
$I_1 = \frac{1}{12} M a^2$ ⟶



$I_2 = \frac{1}{36} M a^2$ ⟶

$I_3 = \frac{1}{9} M a^2$ ⟶

MT 11.2 :



$\rho(\vec{r}) = \rho$ constant mass density

$$I_{ij} = \int \rho(\vec{r}) \left[\delta_{ij} x_k x_k - x_i x_j \right] dv$$

- For convenience, we will use cylindrical coordinates.

$$I_{11} = \rho \int (x_2^2 + x_3^2)\, dx_1 dx_2 dx_3 \qquad \leftarrow \text{in rect. coordinates}$$

$$= \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{x_3 R}{h}} r^2 \cos^2\theta + x_3^2 \; r\,dr\,d\theta\,dx_3 \qquad \leftarrow \text{in cylindrical coordinates}$$

Note: $\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2} \right) d\theta = \pi + \sin 2\theta \Big|_0^{2\pi}$
$= \pi$

- So, doing the θ integral, we have,

$$I_{11} = \rho \int_0^h \int_0^{x_3 R/h} \left[\pi r^2 + 2\pi x_3^2 \right] r\,dr\,dx_3$$

$$= \rho \int_0^h \left[\pi \frac{r^4}{4} \Big|_0^{x_3 R/h} + 2\pi x_3^2 \frac{r^2}{2} \Big|_0^{x_3 R/h} \right] dx_3$$

$$= \rho\pi \int_0^h \left[\frac{1}{4} \left(\frac{R}{h}\right)^4 x_3^4 + \left(\frac{R}{h}\right)^2 x_3^4 \right] dx_3$$

$$= \rho\pi \left[\frac{1}{4} \left(\frac{R}{h}\right)^4 \frac{h^5}{5} + \left(\frac{R}{h}\right)^2 \frac{h^5}{5} \right]$$

$$= \rho\pi \left(\frac{R}{h}\right)^2 \frac{h^5}{5} \left(\frac{1}{4} \left(\frac{R}{h}\right)^2 + 1 \right)$$

- Now, we calculate the total mass M for the cone.

$$M = \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} \rho \, r\,dr\,d\theta\,dx_3$$

$$= 2\pi\rho \int_0^h \left(\frac{R}{h}\right)^2 \frac{x_3^2}{2}\,dx_3$$

$$M = \pi\rho\left(\frac{R}{h}\right)^2 \frac{h^3}{3}$$

- So, $I_{11} = M\left(\frac{3}{5}h^2\right)\left(\frac{1}{4}\left(\frac{R}{h}\right)^2+1\right) = \frac{3}{20}M\left(R^2+4h^2\right)$

- The cone is symmetric with respect to the x_1-x_2 plane,

so,

$$I_{22} = I_{11} = \frac{3}{20}M\left(R^2+4h^2\right)$$

- Now, we calculate I_{33}:

$$I_{33} = \rho \int \left(x_1^2+x_2^2\right) dx_1\,dx_2\,dx_3$$

$$= \rho \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} r^2 \, r\,dr\,d\theta\,dx_3$$

$$= 2\pi\rho \int_0^h \frac{1}{4}\left(\frac{R}{h}\right)^4 x_3^4\,dx_3$$

$$= 2\pi\rho\left(\frac{1}{4}\left(\frac{R}{h}\right)^4\right)\frac{h^5}{5}$$

$$= \pi\rho\left(\frac{R}{h}\right)^2\frac{h^3}{3}\left(\frac{1}{2}\left(\frac{R}{h}\right)^2\cdot h^2\left(\frac{3}{5}\right)\right)$$

$$I_{33} = \frac{3}{10}MR^2$$

$$I_{12} = \rho \int (-x_1 x_2)\, dx_1 dx_2 dx_3$$

$$= -\rho \int_0^h \int_0^{2\pi} \int_0^{\frac{x_3 R}{h}} r^2 \cos\theta \sin\theta \; r\,dr\,d\theta\,dx_3$$

$$= 0 \quad \text{since} \quad \int_0^{2\pi} \cos\theta \sin\theta \, d\theta = 0$$

$$I_{13} = \rho \int (-x_1 x_3)\, dx_1 dx_2 dx_3$$

$$= -\rho \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} r\cos\theta \, x_3 \; r\,dr\,d\theta\,dx_3$$

$$= 0 \quad \text{since} \quad \int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$I_{23} = \rho \int (-x_2 x_3)\, dx_1 dx_2 dx_3$$

$$= -\rho \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} r\sin\theta \, x_3 \; r\,dr\,d\theta\,dx_3$$

$$= 0 \quad \text{since} \quad \int_0^{2\pi} \sin\theta \, d\theta = 0$$

So, we have

$$\underset{\sim}{I} = \begin{pmatrix} \frac{3}{20}M(R^2+4h^2) & 0 & 0 \\ 0 & \frac{3}{20}M(R^2+4h^2) & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

with the cone upright along the x_3 axis and the apex at the origin.

- Now, we calculate the CM:

$$M X_1^{cm} = \int \rho\, x_1\, dx_1\, dx_2\, dx_3$$

$$= \rho \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} r\cos\theta \; r\,dr\,d\theta\,dx_3 = 0$$

$$M X_2^{cm} = X_1^{cm} = 0$$

$$M X_3^{cm} = \int \rho\, x_3\, dx_1\, dx_2\, dx_3$$

$$= \rho \int_0^h \int_0^{2\pi} \int_0^{x_3 R/h} x_3 \; r\,dr\,d\theta\,dx_3$$

$$= 2\pi\rho \int_0^h x_3 \, \frac{1}{2}\left(\frac{R}{h}\right)^2 x_3^2 \, dx_3$$

$$= \pi\rho\left(\frac{R}{h}\right)^2 \frac{h^4}{4}$$

$$M X_3^{cm} = \underbrace{\pi\rho\left(\frac{R}{h}\right)^2 \frac{h^3}{3}}_{M}\left(\frac{3}{4}h\right)$$

$$X_3^{cm} = \frac{3}{4}h$$

- Now, we use the parallel axis theorem to express the $\underline{\underline{I}}^{cm}$ with the origin at the CM.

$$\vec{a} = (0, 0, \tfrac{3}{4}h)$$

$$I^{cm}_{ij} = I_{ij} - M \delta_{ij} a^2 - a_i a_j$$

$$I^{cm}_{11} = I_{11} - M \frac{9}{16} h^2$$

$$= \frac{3}{20} M (R^2 + 4h^2) - \frac{9}{16} M h^2$$

$$= \frac{3}{20} M R^2 + \frac{3}{5} M h^2 - \frac{9}{16} M h^2$$

$$= \frac{3}{20} M R^2 + M h^2 \left(\frac{48}{80} - \frac{45}{80} \right)$$

$$\underline{I^{cm}_{11} = \frac{3}{20} M \left(R^2 + \frac{h^2}{4} \right)}$$

$$\underline{I^{cm}_{22} = I^{cm}_{11}}$$

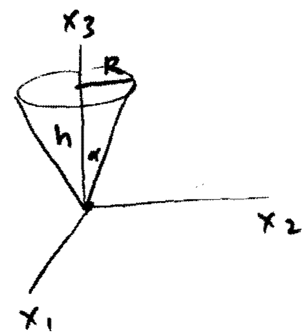
$$I^{cm}_{33} = I_{33} - M \left(\frac{9}{16} h^2 - \frac{9}{16} h^2 \right) = \underline{I_{33} = \frac{3}{10} M R^2}$$

So,

$$\underset{\sim}{I}^{cm} = \begin{pmatrix} \frac{3}{20} M \left(R^2 + \frac{h^2}{4} \right) & 0 & 0 \\ 0 & \frac{3}{20} M \left(R^2 + \frac{h^2}{4} \right) & 0 \\ 0 & 0 & \frac{3}{10} M R^2 \end{pmatrix}$$

Since this is diagonalized already, the diagonal elements are the principle moments.

From MT 11.2, we have computed the moments of inertia of a cone rotating about its symmetry axis and with the origin of a coordinate system at the apex of the cone as shown below:



$\tan\alpha = \frac{R}{h}$, ρ - uniform density

M(total mass) $= \pi\rho\left(\frac{R}{h}\right)^2 \frac{h^3}{3}$

$$\underset{\approx}{I} = \begin{pmatrix} \frac{3}{20}M(R^2+4h^2) & 0 & 0 \\ 0 & \frac{3}{20}M(R^2+4h^2) & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

For this problem, we will have this cone rolling on its side without slipping on a uniform horizontal plane. This is in the space axes. let call this (x_1', x_2', x_3'). The (x_1, x_2, x_3) coordinate system in which we calculated $\underset{\approx}{I}$ is the body axes. These two coordinate systems can be related by the Euler's angles.

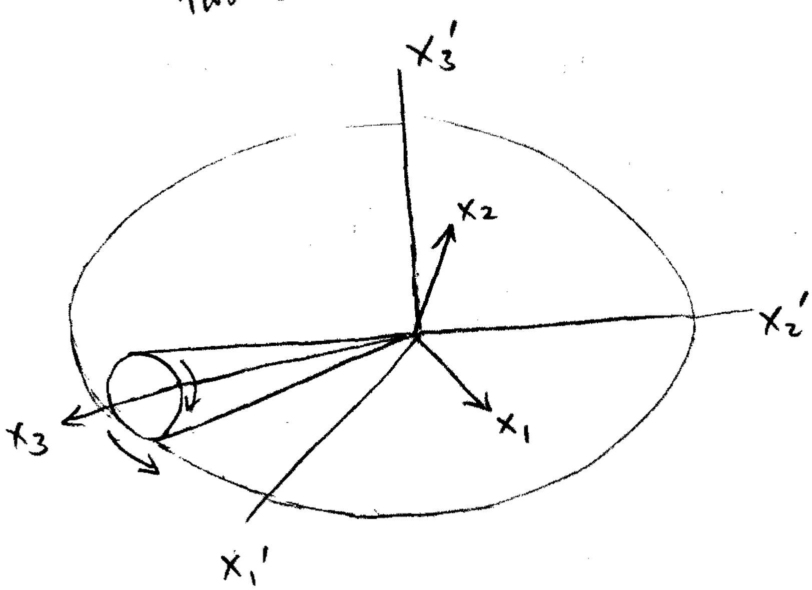
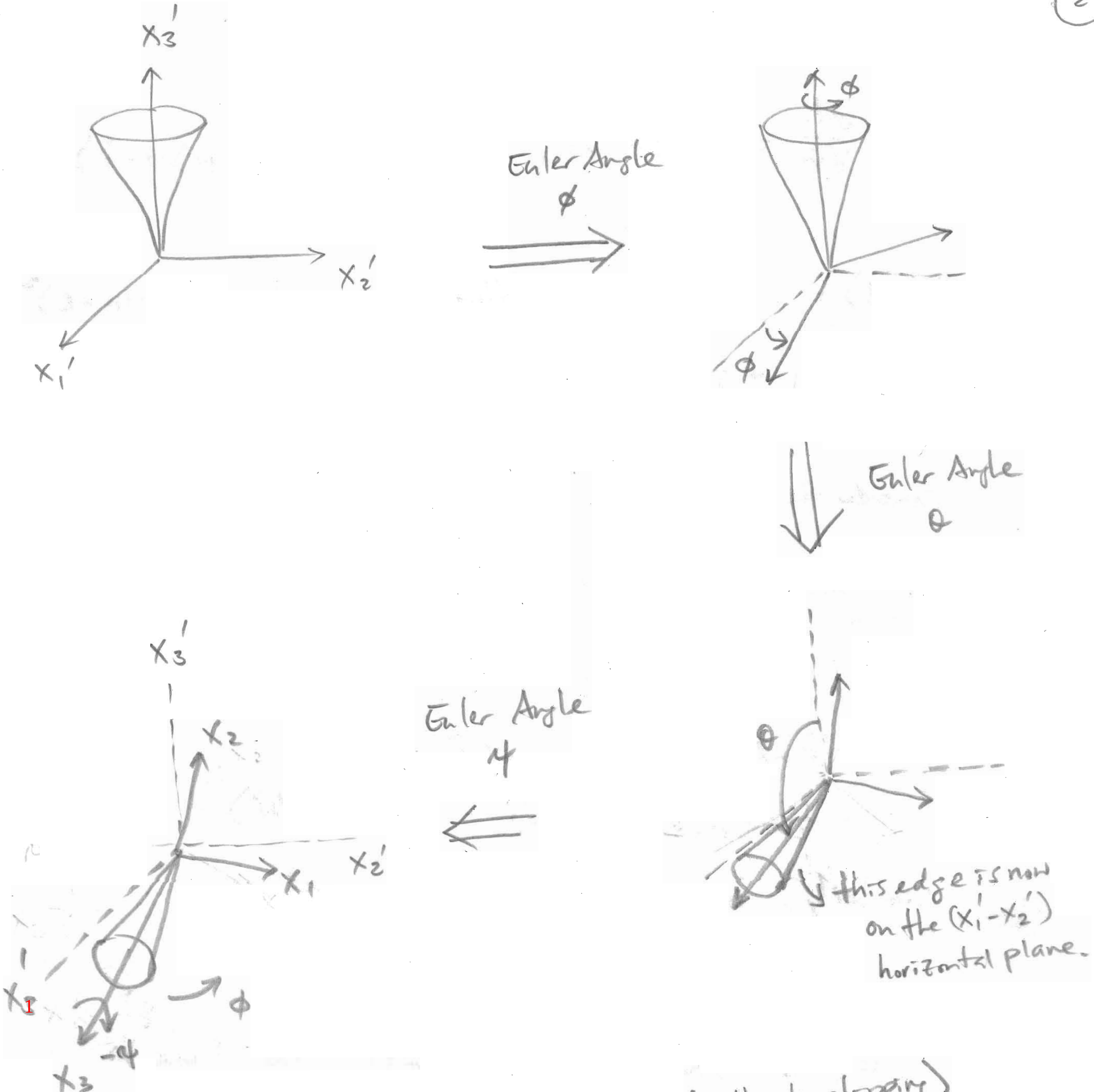


Fig. 1

Cone rolling on its side on a horizontal plane.

(x_1', x_2', x_3') - space axes

(x_1, x_2, x_3) - body axes



- We are given that the cone rolls around (without slipping) the x_3'-axis (space axes) in a time τ.

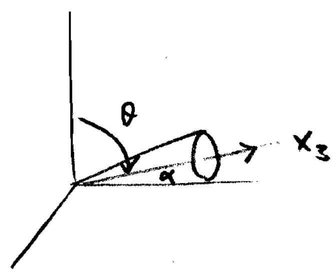
→ We will define $\omega_0 \equiv$ frequency of rolling around the x_3'-axis

where $\omega_0 = \frac{2\pi}{\tau} = \dot{\phi}$

→ And, define $\omega_s \equiv$ frequency of spinning of the cone around its x_3 body axis with the orientation shown in Fig. 1,

$$\underline{\omega_s = -\dot{\psi}}.$$

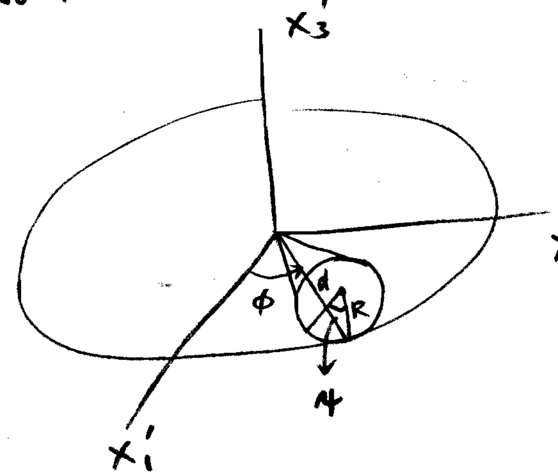
- Together with the geometrical constriant of the cone laying on its edge,

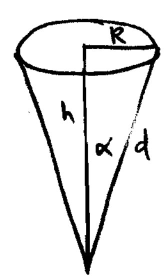


we have the following equations for the Euler's angles:

$$(*)\quad \begin{cases} \theta = \frac{\pi}{2} - \alpha \\ \phi = \omega_0 t \\ \psi = -\omega_s t \end{cases}$$

- Now, we consider the no slip condition:





R/d = sin α

for rolling without slipping, we need to have the following distances to be equal:

$$|\phi d| = |\psi R|$$

For constant rolling speeds, we have

$$(\omega_o t)d = (\omega_s t)R$$

$$\omega_s = \frac{\omega_o}{\sin\alpha}$$

So, the magnitude of the two angular speeds are related.

- With θ well defined, we can write down the components of the angular velocity vector $\vec{\omega}$ for the cone's motion in the body axes using the equations on P.174 (Eq. 4.87):

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

$$\vec{\omega} = \omega_1\hat{x}_1 + \omega_2\hat{x}_2 + \omega_3\hat{x}_3$$

- Substituting what we have for ϕ, θ, ψ & their derivatives, we have

$$\omega_1 = \omega_o \sin(\tfrac{\pi}{2}-\alpha)\sin(-\omega_s t) + 0$$

$$\omega_2 = \omega_o \sin(\tfrac{\pi}{2}-\alpha)\cos(-\omega_s t) + 0$$

$$\omega_3 = \omega_o \cos(\tfrac{\pi}{2}-\alpha) - \omega_s$$

$$\Rightarrow \begin{cases} \omega_1 = -\omega_0 \cos\alpha \sin(\omega_s t) \\ \omega_2 = \omega_0 \cos\alpha \cos(\omega_s t) \\ \omega_3 = \omega_0 \sin\alpha - \dfrac{\omega_0}{\sin\alpha} \end{cases}$$

$$= \omega_0 \left(\frac{\sin^2\alpha - 1}{\sin\alpha} \right)$$

$$= -\frac{\omega_0 \cos^2\alpha}{\sin\alpha} = -\omega_0 \frac{\cos\alpha}{\tan\alpha}$$

- In the body axes, the moments of inertia are given by

$$I_{11} = I_{22} = \frac{3}{20} M (R^2 + 4h^2) \qquad \& \quad I_{12} = I_{13} = I_{23} = 0$$

$$I_{33} = \frac{3}{10} M R^2$$

- Thus, $T = \frac{1}{2} I_{11} \omega_1^2 + \frac{1}{2} I_{22} \omega_2^2 + \frac{1}{2} I_{33} \omega_3^2$

$$= \frac{1}{2}\left(\frac{3}{20} M (R^2 + 4h^2)\right)\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}\left(\frac{3}{10} M R^2\right) \omega_3^2$$

$$= \frac{3}{40} M (R^2 + 4h^2)\left(\omega_0^2 \cos^2\alpha \left(\underbrace{\sin^2(\omega_s t) + \cos^2(\omega_s t)}_{1}\right)\right)$$

$$+ \frac{3}{20} M R^2 \omega_0^2 \frac{\cos^2\alpha}{\tan^2\alpha} \qquad \tan\alpha = \frac{R}{h}$$

$$= \frac{3}{40} M \omega_0^2 \cos^2\alpha \left(R^2 + 4h^2 + 2 \cancel{R^2} \frac{h^2}{\cancel{R^2}} \right)$$

$$T = \frac{3}{40} M \omega_0^2 \cos^2\alpha \left(R^2 + 6h^2 \right)$$

Now, we calculate the angular momentum $\vec{L}$ with respect to the body axes :

$$L_1 = I_{11}\omega_1 = \frac{3}{20}M(R^2+4h^2)\left(-\omega_0\cos\alpha\sin(\omega_s t)\right)$$

$$= -\frac{3}{20}M\omega_0\cos\alpha\,(R^2+4h^2)\sin\left(\frac{\omega_0}{\sin\alpha}t\right)$$

$$L_2 = I_{22}\omega_2 = \frac{3}{20}M(R^2+4h^2)\left(\omega_0\cos\alpha\cos(\omega_s t)\right)$$

$$= \frac{3}{20}M\omega_0\cos\alpha\,(R^2+4h^2)\cos\left(\frac{\omega_0}{\sin\alpha}t\right)$$

$$L_3 = I_{33}\omega_3 = \frac{3}{10}MR^2\left(-\omega_0\frac{\cos\alpha}{\tan\alpha}\right)$$

$$= -\frac{3}{10}M\omega_0\cos\alpha\,R^{\cancel{2}}\left(\frac{h}{\cancel{R}}\right)$$

$$= -\frac{3}{10}M\omega_0\cos\alpha\,Rh$$

$\vec{L}$ can be expressed in the space axes using the inverse Euler transform (Eq. 4-47) on p. 153.

5.26 ①

For a symmetric top without external torques, we have

$I_1 = I_2$ and $\vec{N} = 0$.

So, we have the following Euler's Equation:

$$\begin{cases} I_1 \dot{\omega}_1 = (I_1 - I_3)\,\omega_2 \omega_3 & ① \\ I_1 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1 & ② \\ I_3 \dot{\omega}_3 = 0 & ③ \end{cases}$$

③ $\Rightarrow$ $\omega_3 = \text{const}$

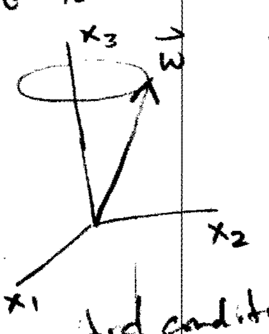
For convenience, let $\Omega = \dfrac{I_3 - I_1}{I_1}\,\omega_3$ or $\omega_3 = \dfrac{I_1}{I_3 - I_1}\,\Omega$

Then, we can write ① & ② as

$$\begin{cases} \dot{\omega}_1 = -\Omega\,\omega_2 \\ \dot{\omega}_2 = \Omega\,\omega_1 \end{cases}$$

The solution for this simple linear ODE is:

$$\omega_1 = A\cos(\Omega t + \delta)$$
$$\omega_2 = A\sin(\Omega t + \delta)$$



$\vec{\omega}$ processes with constant angular velocity Ω about x_3.

where A & δ will be determined by initial conditions.

Equation 4.87 relates the components of the angular velocity vector $\vec{\omega}$ in the body axes to the Euler's angles, ϕ, θ, ψ :

$$\begin{cases} \omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi = A\cos(\Omega t + \delta) & (1)' \\ \omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi = A\sin(\Omega t + \delta) & (2)' \\ \omega_3 = \dot{\phi} \cos\theta + \dot{\psi} = \left(\frac{I_1}{I_3 - I_1}\right)\Omega & (3)' \end{cases}$$

$(1)' \sin\psi + (2)' \cos\psi$:

$$\begin{aligned} &\dot{\phi} \sin\theta \sin^2\psi + \cancel{\dot{\theta} \sin\psi \cos\psi} \\ &+ \dot{\phi} \sin\theta \cos^2\psi - \cancel{\dot{\theta} \sin\psi \cos\psi} \end{aligned} = \begin{aligned} &A\cos(\Omega t + \delta)\sin\psi \\ &+ A\sin(\Omega t + \delta)\cos\psi \end{aligned}$$

$$\Rightarrow \quad \dot{\phi} \sin\theta = A\sin(\Omega t + \delta + \psi) \qquad (4)'$$

$(1)' \cos\psi - (2)' \sin\psi$:

$$\begin{aligned} &\cancel{\dot{\phi} \sin\theta \sin\psi \cos\psi} + \dot{\theta} \cos^2\psi \\ &- \cancel{\dot{\phi} \sin\theta \sin\psi \cos\psi} + \dot{\theta} \sin^2\psi \end{aligned} = \begin{aligned} &A\cos(\Omega t + \delta)\cos\psi \\ &- A\sin(\Omega t + \delta)\sin\psi \end{aligned}$$

$$\Rightarrow \quad \dot{\theta} = A\cos(\Omega t + \delta + \psi) \qquad (5)'$$

For uniform precession, we need to have (3)

$\dot{\theta} = 0$ (no nutation)

& $\dot{\phi}$ = const (constant speed)

(5)' $\Rightarrow$ $\dot{\theta} = A\cos(\Omega t + \delta + \psi) = 0$

$\Rightarrow$ $\Omega t + \delta + \psi = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$

Taking $n=0$ and calling $\psi_0 = -\delta + \pi/2$, we have

$$\underline{\psi = -\Omega t + \psi_0} \qquad (*1)$$

and $\dot{\theta} = 0 \Rightarrow \underline{\theta = \theta_0}$ (*2)

- using (*1) & (*2), (4)' gives

(*3)

$\dot{\phi} = A\sin(\pi/2)/\sin\theta_0$ = const. (as is required by uniform precession).

- To solve for the constant A, we can substitute (*1), (*2), & (*3) back into (3'), we have,

$$\omega_3 = A\frac{\cos\theta_0}{\sin\theta_0} - \Omega$$

$$(\omega_3 + \Omega)\tan\theta_0 = A$$

$$\left(\frac{\cancel{I_1} + I_3 - \cancel{I_1}}{I_3 - I_1}\right)\Omega \tan\theta_0 = A$$

$$A = \Omega \tan\theta_0 \left(\frac{I_3}{I_3 - I_1}\right)$$

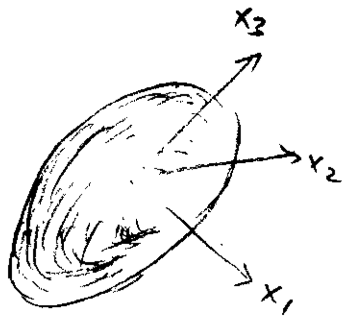
So, $$\dot{\phi} = \Omega \frac{\tan\theta_0}{\sin\theta_0}\left(\frac{I_3}{I_3 - I_1}\right)$$

$$\Rightarrow \quad \underline{\phi = \frac{\Omega}{\cos\theta_0}\left(\frac{I_3}{I_3 - I_1}\right)t + \phi_0} \qquad *3$$

<u>Summary</u> :

$$\begin{cases} \phi = \dfrac{\Omega}{\cos\theta_0}\left(\dfrac{I_3}{I_3 - I_1}\right)t + \phi_0 \\[2ex] \theta = \theta_0 \\[2ex] \psi = -\Omega t + \psi_0 \end{cases}$$

Problem 3 :



$I_1 > I_2 > I_3$

let $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the rotating body. We need to consider three cases :

Case 1 : rotating almost directly along x_1.

If stable, small deviations away from x_1 will not grow in time. To check for this, we can consider the time evolution of $\vec{\omega}$ using the Euler's Equation :

$$\begin{cases} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = 0 & (1) \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = 0 & (2) \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0 & (3) \end{cases}$$

- In this case, we have set initially $\omega_1 \gg \omega_2, \omega_3$.
- Since there is no torque acting on the system, $\vec{L}$ is fixed and $|\vec{\omega}|$ is also fixed.

- $|\vec{\omega}| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} = \text{const}$ ⊛

- Since $\omega_1 \gg \omega_2, \omega_3$, ⊛ implies that

$\underline{\omega_1 \simeq \text{const}}$ to 1st approximation.

Putting this into our Euler's Equations :

- ① does not gives us any useful information
  - it is just a consistency statement saying that $\dot{\omega}_1 \simeq \omega_2 \omega_3 \simeq 0$.
- $\dot{\omega}_2$ & $\dot{\omega}_3$ are not necessary small and we have to 1st order approximation, the following coupled equations :

$$\begin{cases} \dot{\omega}_2 = \dfrac{\omega_1 (I_3 - I_1)}{I_2} \omega_3 \\[2ex] \dot{\omega}_3 = \dfrac{\omega_1 (I_1 - I_2)}{I_3} \omega_2 \end{cases}$$

- Taking the derivative on both sides, we have

$$\begin{cases} \ddot{\omega}_2 = \dfrac{\omega_1 (I_3 - I_1)}{I_2} \dot{\omega}_3 = \dfrac{\omega_1^2 (I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \omega_2 \\[2ex] \ddot{\omega}_3 = \dfrac{\omega_1 (I_1 - I_2)}{I_3} \dot{\omega}_2 = \dfrac{\omega_1^2 (I_1 - I_2)(I_3 - I_1)}{I_3 I_2} \omega_3 \end{cases}$$

- Now, the two equations have decoupled and the constant coefficients on the RHS are the same :

$$\underbrace{\frac{w_1^2\,(\overset{+}{I_1 - I_2})(\overset{-}{I_3 - I_1})}{I_2 I_3} < 0}$$

which is negative since $I_1 > I_2 > I_3$!

- Thus, the time evolution for W_2, W_3 are described by a harmonic equation :

$$\ddot{W}_{2,3} = -\Omega^2 W_{2,3} \quad , \quad \Omega^2 = \frac{W_1^2 (I_1 - I_2)(I_1 - I_3)}{I_2 I_3} > 0$$

with $\underbrace{W_{2,3}(t) \sim e^{\pm i\Omega t}}$.

- Since $W_{2,3}$ will not grow in time, spinning motion along X_1 will remain stable !

<u>Case 2</u> : $W_2 \gg W_1, W_3$

Following a similar argument as case 1, we can arrive at the following 2nd order ODEs for W_1 & W_3 :

$$\begin{cases} \ddot{\omega}_1 = \dfrac{\omega_2(I_2-I_3)}{I_1}\dot{\omega}_3 = \dfrac{\omega_2^2(I_2-I_3)(I_1-I_2)}{I_1 I_3}\omega_1 \\ \ddot{\omega}_3 = \dfrac{\omega_2(I_1-I_2)}{I_3}\dot{\omega}_1 = \dfrac{\omega_2^2(I_1-I_3)(I_2-I_3)}{I_3 I_1}\omega_3 \end{cases}$$

- Since $I_1 > I_2 > I_3$,

$$\gamma^2 = \frac{\omega_2^2 \overset{+}{(I_2-I_3)}\overset{+}{(I_1-I_2)}}{I_1 I_3} > 0$$

$$\Rightarrow \quad \omega_{1,3}(t) \sim e^{\pm\gamma t}$$

$\Rightarrow$ Therefore, $\omega_{1,3}(t)$ might grow in time and spinning initially along X_2 is not stable!

Lastly case 3 : $\omega_3 \gg \omega_1, \omega_2$

In this case, we will have

$$\ddot{\omega}_1 = \frac{\omega_3^2(I_2-I_3)(I_3-I_1)}{I_1 I_2}\omega_1$$

$$\ddot{\omega}_2 = \frac{\omega_3^2(I_3-I_1)(I_2-I_3)}{I_1 I_2}\omega_2$$

Since, $$\frac{w_3^2 \overset{+}{(I_2 - I_3)}\overset{-}{(I_3 - I_1)}}{I_1 I_2} = -\Omega'^2 < 0$$

$w_{1,2}(t) \sim e^{\pm i\Omega' t}$ are harmonic

and spinning along x_3 initially will be

stable !