(11')

1)

$$(\mathbf{A}\times\mathbf{B})_j = \sum_{\ell m} \varepsilon_{j\ell m} A_\ell B_m$$

$$(\mathbf{C}\times\mathbf{D})_k = \sum_{rs} \varepsilon_{krs} C_r D_s$$

$$\begin{aligned}
[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]_i &= \sum_{jk} \varepsilon_{ijk}\left(\sum_{\ell m} \varepsilon_{j\ell m} A_\ell B_m\right)\left(\sum_{rs} \varepsilon_{krs} C_r D_s\right) \\
&= \sum_{jk\ell mrs} \varepsilon_{ijk}\, \varepsilon_{j\ell m}\, \varepsilon_{krs} A_\ell B_m C_r D_s \\
&= \sum_{j\ell mrs} \varepsilon_{j\ell m}\left(\sum_{k} \varepsilon_{ijk}\, \varepsilon_{rsk}\right) A_\ell B_m C_r D_s \\
&= \sum_{j\ell mrs} \varepsilon_{j\ell m}\left(\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr}\right) A_\ell B_m C_r D_s \\
&= \sum_{j\ell m} \varepsilon_{j\ell m}\left(A_\ell B_m C_i D_j - A_\ell B_m D_i C_j\right) \\
&= \left(\sum_{j\ell m} \varepsilon_{j\ell m} D_j A_\ell B_m\right) C_i - \left(\sum_{j\ell m} \varepsilon_{j\ell m} C_j A_\ell B_m\right) D_i \\
&= (\mathbf{ABD})C_i - (\mathbf{ABC})D_i
\end{aligned}$$

Therefore,

$$\boxed{[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})] = (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D}}$$

2)

First $(\mathbf{B}\times\mathbf{C})_i = \sum_{jk} \varepsilon_{ijk} B_j C_k$. Then,

$$\begin{aligned}
\left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right]_\ell &= \sum_{mn} \varepsilon_{\ell mn} A_m (B\times C)_n = \sum_{mn} \varepsilon_{\ell mn} A_m \sum_{jk} \varepsilon_{njk} B_j C_k \\
&= \sum_{jkmn} \varepsilon_{\ell mn}\, \varepsilon_{njk}\, A_m B_j C_k = \sum_{jkmn} \varepsilon_{\ell mn}\, \varepsilon_{jkn}\, A_m B_j C_k \\
&= \sum_{jkm} \left(\sum_n \varepsilon_{lmn}\, \varepsilon_{jkn} \right) A_m B_j C_k \\
&= \sum_{jkm} \left(\delta_{jl}\delta_{km} - \delta_{k\ell}\delta_{jm} \right) A_m B_j C_k \\
&= \sum_m A_m B_\ell C_m - \sum_m A_m B_m C_\ell = B_\ell \left(\sum_m A_m C_m \right) - C_\ell \left(\sum_m A_m B_m \right) \\
&= (\mathbf{A}\cdot\mathbf{C}) B_\ell - (\mathbf{A}\cdot\mathbf{B}) C_\ell
\end{aligned}$$

Therefore,

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}}$$

(9)

4.15

$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$. Write its components out in the space set of axes in terms of the Euler angles.

$\vec{\omega}_\phi$, $\vec{\omega}_\theta$, and $\vec{\omega}_\psi$ are the three successive infinitesimal rotations with angular velocities

$$\vec{\omega}_\phi = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix}, \quad \vec{\omega}_\theta = \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\omega}_\psi = \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

along $\hat{z}$ — along the line of nodes $\hat{\xi}$ — along $\hat{z}'$

(1) $\vec{\omega}_\phi$ is already along the space axis, no change is needed.

(2) $\vec{\omega}_\theta$ is along the line of nodes. To express it back to the space axes, rotate it by "$-\phi$" using the D-matrix (inverse of D, D^{-1}):

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot\theta\cos\phi \\ \dot\theta\sin\phi \\ 0 \end{pmatrix} = \vec{\omega}_{\theta s}$$

Note: To get from the $\vec{\xi}$ set of intermediate axes back to $\vec{x}$, we formally need $(CD)^{-1} = D^{-1}C^{-1}$. However, since C^{-1} simply rotates x-component of a vector, it does not affect $\vec{\omega}_\theta$. It suffices to use D^{-1} only.

(2)

(3) $\vec{\omega}_\psi$ is along the body axes $\hat{z}'$. To express it back to the space axes, rotate it by the inverse $(BCD)^{-1} = D^{-1}C^{-1}B^{-1}$. Again, since B^{-1} rotates the z-component of a vector, it suffices to consider $D^{-1}C^{-1}$ only:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi\cos\theta & \sin\phi\sin\theta \\ \sin\phi & \cos\phi\cos\theta & -\cos\phi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

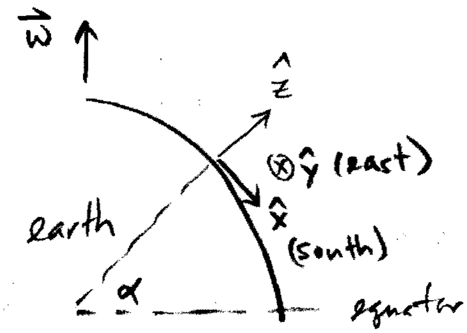
$$= \begin{pmatrix} \dot{\psi}\sin\phi\sin\theta \\ -\dot{\psi}\cos\phi\sin\theta \\ \dot{\psi}\cos\theta \end{pmatrix} = (\vec{\omega}_\psi)_s$$

Thus, in the space axes,

$$(\vec{\omega})_s = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\ \dot{\theta}\sin\phi - \dot{\psi}\cos\phi\sin\theta \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix}$$

4.21

Here is the geometry of the coordinate system:

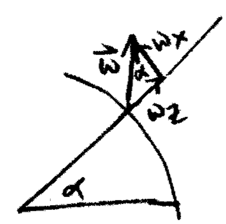


- Considering the Coriolis effect alone (ignoring the centrifugal force), the equation of motion for a particle moving with velocity $\vec{v}$ is given by:

$$-2m(\vec{\omega}\times\vec{v}) - mg\hat{z} = m\vec{a}$$

$$\Rightarrow \qquad \dot{\vec{v}} = -2(\vec{\omega}\times\vec{v}) - g\hat{z} \qquad (*)$$

- For the coordinates chosen, we have



$$\vec{\omega} = -\omega\cos\alpha\,\hat{x} + \omega\sin\alpha\,\hat{z}$$

- Then, $$\vec{\omega}\times\vec{v} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_x & 0 & \omega_z \\ v_x & v_y & v_z \end{pmatrix}$$

$$= -\omega_z v_y\,\hat{x} + (\omega_z v_x - \omega_x v_z)\hat{y} + \omega_x v_y\hat{z}$$

So that, in component form, we can write :

$$(*) \quad \begin{cases} \dot{v}_x = 2\omega \sin\alpha \, v_y \\ \dot{v}_y = -2\omega(\cos\alpha \, v_z + \sin\alpha \, v_x) \\ \dot{v}_z = 2\omega\cos\alpha \, v_y - g \end{cases}$$

- Obviously, to zero order in ω, we have the simple free fall situation with

$$\dot{v}_x^{(0)} = 0 \; ; \quad \dot{v}_y^{(0)} = 0 \; ; \quad \dot{v}_z^{(0)} = -g$$

Case 1: with the initial condition (throwing straight up with velocity $v(0) = v_0 \hat{z}$, we have

$$v_z^{(0)} = v_0 - gt$$

and $z^{(0)}(t) = z_0 + v_0 t - \frac{1}{2} g t^2$

And, $v_x^{(0)} = 0$ and $v_y^{(0)} = 0$

- Now, we will consider the perturbative solution upto $O(\omega)$ by substituting $\vec{v}^{(0)}$ into the RHS of (*) :

$$\dot{v}_x^{(1)} = 0$$

$$\dot{v}_y^{(1)} = -2\omega\cos\alpha\,(v_0 - gt)$$

$$\dot{v}_z^{(1)} = -g$$

- Up to $O(\omega)$, the Coriolis force affect only the y-dir only.

(From now on, I will subpress the order notation.)

$$\dot{v}_y = -2\omega\cos\alpha(v_0 - gt)$$

I.C.

So, $v_y = -2\omega\cos\alpha\, v_0 t + g\,\omega\cos\alpha\, t^2$ $\quad (v_y(0) = 0)$

$y = -\omega\cos\alpha\, v_0 t^2 + \frac{1}{3} g\,\omega\cos\alpha\, t^3$ $\quad (y(0) = 0)$

$v_z = v_0 - gt$; $z = v_0 t - \frac{1}{2} g t^2$ $\quad (z(0) = 0)$

- The particle will reach its maximum height h when $v_z = 0$ at $t = v_0/g$ (top)

& $h = \frac{v_0^2}{g} - \frac{1}{2} g \left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}$

- It will fall back down to the ground at

$0 = v_0 - \frac{1}{2} g t \Rightarrow t = \frac{2v_0}{g}$ (ground again)

- At this time, the y deflection is :

$$y = -\omega\cos\alpha\, v_0 \left(\frac{2v_0}{g}\right)^2 + \frac{1}{3} g\,\omega\cos\alpha \left(\frac{2v_0}{g}\right)^3$$

$$= \omega\cos\alpha \frac{v_0^3}{g^2}\left(-4 + \frac{8}{3}\right) = -\frac{4}{3}\omega\cos\alpha \frac{v_0^3}{g^2}$$

The negative sign indicates that the deflection is to the west.

Case 2:

Now, we consider the other situation when the particle is released at h at rest instead:

I.C. : $z(0) = h$, $v_z(0) = v_x(0) = v_y(0) = 0$

From the z equation, we can calculate the time when it hits the ground.

$$z = z_0 + \cancel{v_z(0)t}^{\,0} - \frac{1}{2}gt^2$$

$$0 = h - \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

The y-dir equation is,

$$\dot{v}_y = -2\omega\cos\alpha\, v_z = -2\omega\cos\alpha\,(-gt) \qquad \begin{cases} \dot{v}_z = -g \\ v_z = -gt + \cancel{v_z(0)}^{\,0} \end{cases}$$

$$v_y = \omega\cos\alpha\, g t^2$$

$$y = \frac{1}{3}\omega\cos\alpha\, g t^3$$

At $t = \sqrt{\frac{2h}{g}}$, $y = \frac{1}{3}\omega\cos\alpha\, g\left(\frac{2h}{g}\right)^{3/2}$

In terms of v_0 from the other result, we have

$$h = \frac{v_0^2}{2g}$$

So, $$\frac{2h}{g} = \frac{v_0^2}{g^2}$$

Then, $$Y = \frac{1}{3}\omega\cos\alpha\, g\left(\frac{v_0^2}{g^2}\right)^{3/2} = \frac{1}{3}\omega\cos\alpha\, \cancel{g}\,\frac{v_0^3}{g^{\cancel{3}2}}$$

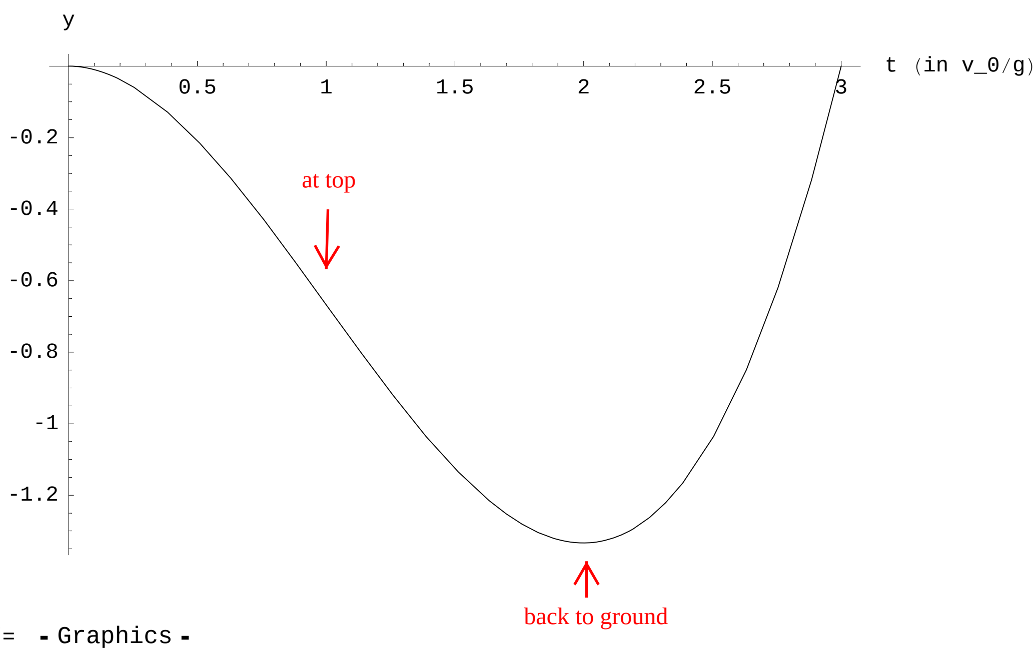
$$= \frac{1}{3}\omega\cos\alpha\,\frac{v_0^3}{g^2}$$

- The + sign indicates that the deflection is to the east!

And it has a magnitude which is $\frac{1}{4}$ of the other result.

$$\Rightarrow \quad Y_{case\,1} = -4\,Y_{case\,2}$$

```
In[19]:= Plot[{-t^2 + t^3 / 3}, {t, 0, 3}, AxesLabel -> {"t (in v_0/g)", "y"}]
```

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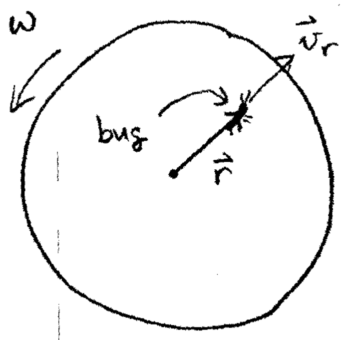
Out[19]= - Graphics -
```

Goldstein P 4.21: Plot of the 1st order deflection in y with the ball initially thrown straight up vertically. Time is in units of v_0/g. The ball reaches the top of its trajectory when t=v_0/g and it falls back down to the ground at t=2v_0/g. The deflection is initially westward (-y) and it becomes eastward (+y) only after t>=3v_0/g.

Note: In Case 1, v_y is non-zero (in fact still negative --> going westward) at the top of trajectory so that it continues to move westward as it comes back down while its v_y continues to decrease eventually going back to zero at ground level. However, as a result, the trajectory still has a westward deflection. In fact, it has its maximum westward deflection when it comes back to the ground (see above figure).

In Case 2, the particle is dropped from REST with ALL its velocity components equal to zero.

4.24



Given:

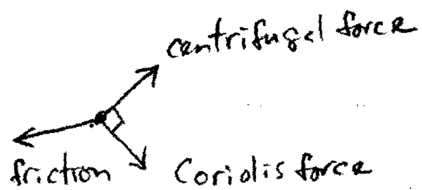
$\omega = 3.0$ radian/s (out of page)

$\vec{v}_r = v_r \hat{r}$, $v_r = 0.5$ cm/s

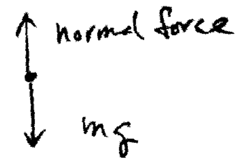
$\mu = 0.30$

- Free body digram in the rotating frame:

Top view



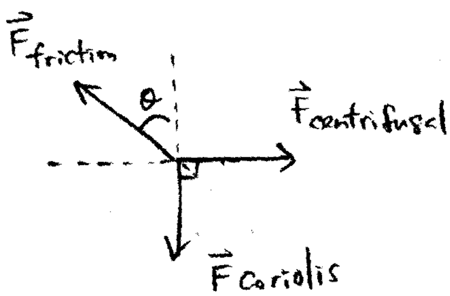
Side view



- In the rotating frame, we have the following equation of motion.

$$\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\,\vec{a}_r = 0$$

where $\vec{F}$ contains gravity, the normal force and friction.



☆ At the point where the bug can barely hold on, the friction force will on the threshold in balancing the centrifugal and the coriolis forces as shown to the left.

— $\vec{\omega}$ is $\perp$ to the x-y plane, so

$$|F_{centrif}| = m\omega^2 r$$

$$|F_{coriolis}| = 2m\omega v_r$$

And, $|F_{friction}| = \mu m g$

— so, we have

$$m\omega^2 r = \mu m g \sin\theta \quad (1)$$

$$2m\omega v_r = \mu m g \cos\theta \quad (2)$$

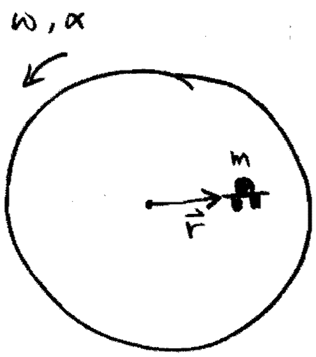
$$(1)^2 + (2)^2 \Rightarrow \cancel{m^2}\omega^4 r^2 + 4\cancel{m^2}\omega^2 v_r^2 = \mu^2 \cancel{m^2} g^2$$

$$r^2 = \frac{\mu^2 g^2 - 4\omega^2 v_r^2}{\omega^4}$$

$$r = \left[\frac{(0.3)^2 (9.81\ m/s^2)^2 - 4(3.0\ rad/s)^2 (0.5 \times 10^{-2}\ m/s)^2}{(3.0\ rad/s)^4}\right]^{1/2}$$

$$r = 32.7\ cm$$

4.25



- both $\vec{\alpha}$ and $\vec{\omega}$ are out of the page
- $\vec{\alpha} = 0.02\,\text{rev}/s^2 = 2\pi(0.02)\,\text{rad}/s^2$ (up)
- Constant angular acceleration,

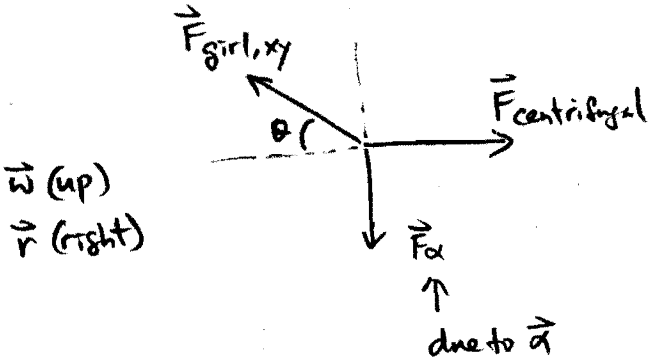
$$\omega = \omega_0 + \alpha t = \alpha t \text{ (up)} \ (\omega_0 = 0)$$

- In the rotating frame, we have the following equation of motion:

since $\vec{v}_r = 0$

$$\vec{F}_{eff} = \underset{\substack{\uparrow \\ \text{this includes gravity} \\ \oplus\ \vec{F}_{girl}}}{\vec{F}} - 2m\cancel{(\vec{\omega} \times \vec{v}_r)}^{\,0} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m(\underset{\substack{\downarrow \\ \dot{\vec{\omega}} = \vec{\alpha}}}{\dot{\vec{\omega}}} \times \vec{r}) = m\vec{a}_r = 0.$$

- Since $\vec{\omega}, \vec{\alpha}$ are $\perp$ to the x-y plane, the $\hat{z}$ dynamic is trivial : $mg = F_{girl,z}$
- On the x-y plane, we have the following free body diagram:



$\vec{F}_{girl,xy}$ - the projection of $\vec{F}_{girl}$ on the x-y plane.

- This is the relevant one since we don't really care about the trivial $F_{girl,z}$ in the $\hat{z}$ direction.

→ We need to find the magnitude of $\vec{F}_{girl,xy}$ & θ.

To balance the forces, we have

$$F_{girl}\cos\theta = F_{cent} = m\omega^2 r \quad \text{—(1)}$$

$$F_{girl}\sin\theta = F_{\alpha} = m\alpha r \quad \text{—(2)}$$

$(1)^2 + (2)^2 \Rightarrow$

$$F_{girl} = \left[m^2\omega^4 r^2 + m^2\alpha^2 r^2\right]^{1/2}$$

$$= mr\left(\alpha^4 t^4 + \alpha^2\right)^{1/2} \qquad \omega = \alpha t$$

$$= mr\alpha\left(\alpha^2 t^4 + 1\right)^{1/2}$$

$$= (3.0\ \text{kg})(7.0\ \text{m})(2\pi\cdot 0.02\ \text{rad/s}^2)\left[(2\pi\cdot 0.02)^2(6.0)^4 + 1\right]^{1/2}$$

$$= (2.64)(4.63)\ \text{N}$$

$$= \underline{12.2\ \text{N}}$$

$$\frac{(2)}{(1)} \Rightarrow \tan\theta = \frac{\cancel{m}\alpha\cancel{r}}{\cancel{m}\omega^2\cancel{r}} = \frac{\alpha}{(\alpha t)^2} = \frac{1}{\alpha t^2}$$

$$= \frac{1}{(2\pi)(0.02)\ \text{rad/s}^2\cdot(6.0\text{s})^2} = 0.2210$$

$\underline{\theta = 12.5°}$ $(167.5°$ cc from $\vec{r})$