other expressions:

\[ Pe(n) = \frac{1}{2^{n!}} \frac{d^n}{d\eta^n} (\eta^2 - 1)^n \]

**Rogers's Formula**

\[ Pe(n) = \sum_{k=0}^{[\theta]} (-1)^k \frac{(2\ell-k)!}{2^k k! (\ell-k)! (\ell-2k)!} \eta^{\ell-2k} \]

**Alternating Series** – delicate cancellation between successive terms.

(Not accurate for large \( \ell \))

- Single \( \ell \) up to 60 or so
- Double \( \ell \) up to 15 or 18.

**A Stable Recurrence for Associated Legendre Polynomial (m ≠ 0)**

\[ (\ell - m) Pe^m = \eta (2\ell - 1) Pe^{m-1} - (\ell + m - 1) Pe^{m-2} \]

**Note:** 0 ≤ m < \( \ell \) ·

\( \ell \) = z-component of angular momentum L

\( \eta \) this recursion is for m ≠ 0

(for \( Pe, m \) see next page)
useful because closed-form expression for starting value,

\[ P_m^n = (-1)^m (2m-1)!! (1 - r^2)^m/2 \]

for odd integers less than \( n \).

using \( \Theta \) & \( P_{m-1} = 0, \)

\[ P_m^{n+1} = \eta (2m+1) P_m^n \]

(formal definition of \( m \))

these forms two starting values for \( \Theta \) to

- Go higher in \( r \)

\[ Y_{\ell, m}(\theta, \phi) = (-1)^m \sqrt{\frac{(2m)!}{4\pi (2\ell + 1)!}} r^m \cos^\ell \phi \cdot P_{\ell, m}(\cos \theta) \]

\[ \int_0^{2\pi} \int_0^\pi Y_{\ell, m}^* (\theta, \phi) Y_{\ell', m'} (\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{\ell\ell'} \delta_{mm'} \]

\[ -m's \rightarrow Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell, m}(\theta, \phi) \]
Bessel Function of Integer Order

\[ \nabla^2 \Phi = 0 \quad \text{in cylindrical coordinates} \]

\[ \nabla^2 = \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \]

Separation of Variables again:

\[ \Phi(p, \phi, z) = R(p) Q(\phi) \Xi(z) \]

\[ \Rightarrow \quad \frac{d^2 \Xi}{dz^2} - k^2 \Xi = 0 \]

\[ \frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \]

\[ \frac{d^2 R}{dp^2} + \frac{1}{p} \frac{dR}{dp} + (k^2 - \frac{\nu^2}{p^2}) R = 0 \]

\[ \Rightarrow \quad \Xi(z) = e^{\pm k z} \]

\[ Q(\phi) = e^{\pm i \nu \phi} \]
Separation of variables

\[
\frac{\partial^2}{\partial \rho^2} R \Omega Z + \frac{1}{\rho} \frac{\partial}{\partial \rho} (R \Omega Z) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} R \Omega Z + \frac{\partial^2}{\partial Z^2} R \Omega Z = 0
\]

\[
\frac{1}{R \Omega Z} \left( R \Omega \left[ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 R}{\partial \phi^2} \right) + \frac{1}{\rho^2} \frac{\partial^2 Q}{\partial \phi^2} + \frac{1}{\Omega} \frac{\partial^2 Z}{\partial Z^2} = 0
\]

\[
\frac{1}{R} \left[ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial R}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 Q}{\partial \phi^2} = -K^2
\]

\[
\frac{\partial^2 Z}{\partial Z^2} - K^2 Z = 0
\]

\[
\frac{1}{R} \left[ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial R}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 Q}{\partial \phi^2} = -K^2
\]

\[
\frac{\rho^2}{K} \left[ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R}{\partial \rho} \right] + K^2 \rho^2 Z^2 + \frac{1}{\Omega} \frac{\partial^2 Q}{\partial \phi^2} = 0
\]
\[
\frac{\partial^2 Q}{\partial \phi^2} + \nu^2 Q = 0
\]

\[
\left\{ \begin{aligned}
\frac{\rho^2}{R} \left[ \frac{\partial^2 K}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial K}{\partial \rho} \right] + K R^2 &= \nu^2 \\
\frac{\partial^2 K}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial K}{\partial \rho} + \left( K^2 - \frac{\nu^2}{R^2} \right) R &= 0
\end{aligned} \right.
\]
Radial equation: \( x = kp \) rescale

\[
d^2R \over dx^2 + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \quad \text{(See next page)}
\]

(\textbf{Bessel function of order } \nu). \\

\textbf{Solutions are the Bessel functions}

\[
J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^\infty \frac{(-1)^j}{j! \Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}
\]

A linear independent solution is \( Y_\nu(x) \) \((\nu \in \mathbb{Z})\)

\[
Y_\nu(x) = \lim_{\nu \to \nu} \frac{J_\nu(x) \cos(\nu \phi) - J_{-\nu}(x)}{\sin(\nu \phi)}
\]

(\textbf{Note: since } \mathcal{J} \text{ is periodic in } \phi \text{, then } \nu \text{ has to be integer})

\[\Rightarrow \text{Bessel Functions of integer order} \]

\[
0 < x < \nu : \\
\begin{cases} 
J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{1}{2x}\right)^\nu & \nu \geq 0 \\
Y_\nu(x) \sim \frac{2}{\pi} \ln(x) \\
K_\nu(x) \sim \frac{\Gamma(\nu)}{2^\nu} \left(\frac{1}{2x}\right)^\nu & \nu > 0
\end{cases}
\]

[If \( \nu \notin \mathbb{Z} \), \( J_\nu \) \& \( J_{-\nu} \) are linearly independent.]
check for self adjoint and compare with table

\[
\begin{align*}
\frac{d}{dx} \left[ x \frac{du}{dx} \right] - \frac{y^2}{x} u + a^2 x u &= 0 & \text{from table} \\
X u + \lambda w(x) u &= 0 \\
x \frac{d^2 u}{dx^2} + \frac{du}{dx} - \frac{y^2}{x} u + a^2 x u &= 0 \\
\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \left( 1 - \frac{y^2}{x^2} \right) u &= 0 & a = 1
\end{align*}
\]

↑

Bessel's Equation
For \( x \gg 1 \), both \( J_n(x) \) and \( Y_n(x) \) look like sine and cosine with decays \( x^{-1/2} \):

\[
J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{1}{2} y \tau - \frac{1}{4} \zeta \right)
\]

\[
Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{1}{2} y \tau - \frac{1}{4} \zeta \right)
\]

For \( x \approx 0 \):

\[
J_n(x) \sim \frac{0.4473}{x^{1/2}}
\]

\[
Y_n(x) \sim -\frac{0.7748}{x^{1/2}}
\]

Orthogonality Condition (with respect to b.c. \( R(0) = 0 \)):

\[
\int_0^a \rho J_n(x \sin \frac{\rho}{a}) J_n(x \sin \frac{L}{a}) d\rho = \frac{a^2}{2} J_{n+1}(x \sin a) \sin \frac{\pi n}{2}
\]

\( n = 0 \) is the only solution.

For given \( n \),

\( \{J_n(x \sin a)\} \) forms a complete set.

\( x \sin a \) are roots of \( J_n(x) \) such that \( J_n(x \sin a) = 0 \).

Graph: \( J_n(x) \) has infinitely many roots!
Recursion Relation:

\[
J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)
\]
\[
Y_{n+1}(x) = \frac{2n}{x} Y_n(x) - Y_{n-1}(x)
\]

\( \text{this is unstable} \) (upward)

\( \text{this is stable!} \) (upward)

1. Need \( J_0, J_1, Y_0, Y_1 \), to start
2. Use recurrence to get others

\( Y_n \)'s are OK
\( J_n \)'s are unstable

Miller's Algorithm

If a recurrence relation is exponentially unstable in one direction, then the reverse iteration is stable! \[ X_n = \frac{1}{2} X_{n+1} \mod 1 \]

The reverse iteration will converge to the desired seq. of the times an unknown \( \text{normalization factor} \) (with arbitrary i.e.)

(Note: the recursion relation is invariant with respect to an overall factor.)
One possible normalization formula for Bessel functions

\[ l = J_0(x) + 2J_2(x) + 2J_4(x) + J_6(x) \]

Steps: (Press)

1. Start with some arbitrary values as our seed for the downward iteration.
   \[ c^2 \approx \# \text{ of SF} \]
   \[ N + \sqrt{K} \approx \text{convergent time} \]
   Order which we want

2. Downward iterate to \( J_0 \) or \( J_1 \).
3. Calculate \( J_0 \) or \( J_1 \) (later).
4. Normalized with \( \textcircled{2} \)!

For \( Y_0(x) \), upward is stable.

Steps: ① Calculate \( Y_0 \) or \( Y_1 \) (later)
       ② Upward iterate to desired \( n \)!
Evaluation of $J_0, J_1, Y_0, Y_1$:

1. For $x < x_0 = \varepsilon$ (arbitrary small value of $x$),
   - approximate $J_0(x)$ and $J_1(x)$ by rational functions in $x$:
     \[ \frac{P(x)}{Q(x)} \]
     see NR
   - approximate the regular part of $Y_0(x)$ and $Y_1(x)$ by rational functions:
     \[ Y_0(x) = \frac{2}{\pi} J_0(x) \ln(x) \]
     \[ Y_1(x) = \frac{2}{\pi} [J_1(x) \ln(x) - \frac{1}{x}] \] (from asymptotic)

2. For $x > x_0$, use approx. look like sine & cosine
   \[ J_{1,0}(x) = \sqrt{\frac{2}{\pi x}} \left[ P_{1,0} \left( \frac{8}{x} \right) \cos(\pi x n) - \right. \]
   \[ Q_{1,0} \left( \frac{8}{x} \right) \sin(\pi x n) \]
   \[ Y_{1,0}(x) = \sqrt{\frac{2}{\pi x}} \left[ P_{1,0} \left( \frac{8}{x} \right) \sin(\pi x n) + Q_{1,0} \left( \frac{8}{x} \right) \right. \]
   \[ \cos(\pi x n) \]
\[ X_n = x - \frac{2n+1}{n} \]

& Po, P_1, Q_0, Q_1 are polynomials.

\[ \begin{cases} \text{even} \\ \text{odd} \end{cases} \]

\[ \rightarrow \text{parameters are given in Press, Hart} \]

\underline{Spherical Bessel Functions}

\[ \frac{d}{dx} \left( r^2 \frac{d}{dr} \right) R(r) + (kr)^2 R(r) = \ell (\ell + 1) R(r) \]

from \[ \Delta^2 \psi + k^2 \psi = 0 \]

\[ J_n(x) = \sqrt{\frac{2}{\pi x}} J_{n+\frac{1}{2}}(x) \]

\[ Y_n(x) = \sqrt{\frac{2}{\pi x}} Y_{n+\frac{1}{2}}(x) \]

\[ \text{Bessel functions of fractional order (later)} \]
Error Function

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \text{erf}(x) \]

Note: the normalization is taken so that

\[ \text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1 \]

Rational Polynomial approximation

1. \[ t = \frac{1}{1+px} \]

\[ \text{erf}(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2} \]

\[ P, a_1, a_2, a_3 \rightarrow P.147 \quad |e| \leq 2.5 \times 10^{-5} \]
\[
\text{erf}(x) = 1 - (b_1 x + b_2 x^3 + b_3 x^5 + b_4 x^7 + b_5 x^9) e^{-x^2}
\]

\[P', b_1, b_2, b_3, b_4, b_5 \Rightarrow |E| \leq 1.5 \times 10^{-7}\]

---

**Gamma Function**

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt
\]

Integrating by parts,

\[
\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt
\]

\[
= -e^{-t} t^z \bigg|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} \, dt
\]

\[
= z \Gamma(z)
\]

For \(z\) integers:

\[
\Gamma(n) = (n-1)!
\]
Polynomial approximations

\[ \Gamma(1+x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} \quad 0 < x < 1 \]

\[ |\varepsilon| \leq 5 \times 10^{-5} \]
\[ |\varepsilon| \leq 3 \times 10^{-7} \]

Use \( \Gamma(z+1) = z \Gamma(z) \) to get other values.

For \( z \to 0 \)

Special values of \( \Gamma(z) \):

\[ z = 1 \implies 0! = 1 \]
\[ z = 0 \implies (-1)! = 0 \]
\[ z = -n \implies n! = 0 \]