other expressions:

$$- \quad P_\ell(\eta) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{d\eta^\ell} (\eta^2-1)^\ell$$

Rodrigues Formula

$$- \quad P_\ell(\eta) = \sum_{k=0}^{[\ell/2]} (-1)^k \frac{(2\ell-2k)!}{2^\ell k!(\ell-k)!(\ell-2k)!} \eta^{\ell-2k}$$

$$[\tfrac{\ell}{2}] = \begin{cases} j & \text{if } \ell = 2j \\ j-1 & \text{if } \ell = 2j-1 \end{cases}$$

alternating series - delicate cancellations between successive terms.

(Not accurate for large ℓ)

single → ℓ up to 6 or 8

double → ℓ up to 10 to 18.

A stable recurrence for associated Legendre Polynomial ($m \neq 0$, >0)

$$(**) \quad (\ell-m) P_\ell^m = \eta (2\ell-1) P_{\ell-1}^m - (\ell+m-1) P_{\ell-2}^m$$

this recursion is for $m \geq 0$ (for $Y_{\ell,-m}$ see next page)

Note: $0 \leq m \leq \ell$

L_z z-component of angular momentum L

useful because closed form expression for starting value,

$$P_m^m = (-1)^m (2m-1)!!\,(1-\eta^2)^{m/2} \quad (*)$$

($n!!$ odd integers less than n.)

using $(**)$ & $P_{m-1}^m = 0$,

$$P_{m+1}^m = \eta(2m+1) P_m^m$$

(for a particular choice of m)

– these forms two starting values for $(**)$ to go higher in ℓ!

e.g. $P_0^0 \to P_1^0, P_2^0, \ldots \quad P_\ell^m$

$P_1^1 \to P_2^1, P_3^1, \ldots \quad -\ell \le m \le \ell$

$P_2^2 \to P_3^2, P_4^2, \ldots$

Know how to generate from above

Note: normal ordering

P_0^0

P_1^0

$P_1^1 \quad P_1^{-1}$

P_2^0

$P_2^1 \quad P_2^{-1}$

$P_2^2 \quad P_2^{-2}$

$\vdots$

Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, e^{im\phi} P_{\ell,m}(\cos\theta)$$

↑ normalization

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell m}^*(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \delta_{\ell\ell'}\delta_{mm'}$$

$\boxed{-m\text{'s}} \to Y_{\ell,m}^*(\theta,\phi) = (-1)^m Y_{\ell,-m}(\theta,\phi)$

Bessel Function of Integer Order

$\nabla^2 \varphi = 0$ in cylindrical coordinates.

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Separation of variables again:

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

$$\Rightarrow \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0$$

$$\Rightarrow \quad \begin{cases} Z(z) = e^{\pm k z} \\ Q(\phi) = e^{\pm i \nu \phi} \end{cases}$$

Separation of variables

$$\frac{\partial^2}{\partial\rho^2} RQZ + \frac{1}{\rho}\frac{\partial}{\partial\rho} RQZ + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} RQZ + \frac{\partial^2}{\partial z^2} RQZ = 0$$

$$\frac{1}{RQZ}\left(QZ\left[\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}R\right] + \frac{1}{\rho^2}\frac{\partial^2 Q}{\partial\phi^2} RZ + RQ\frac{\partial^2 Z}{\partial z^2} = 0\right)$$

$$\underbrace{\frac{1}{R}\left[\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2}}_{-K^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{+K^2} = 0$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 Z}{\partial z^2} - K^2 Z = 0 \\[2ex] \dfrac{1}{R}\left[\dfrac{\partial^2 R}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho}\right] + \dfrac{1}{\rho^2}\dfrac{1}{Q}\dfrac{\partial^2 Q}{\partial\phi^2} = -K^2 \end{cases}$$

$$\cancel{\frac{\rho}{R}\frac{\partial^2 R}{\partial\rho^2}}$$

$$\underbrace{\frac{\rho^2}{R}\left[\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho}\right] + K^2\rho^2}_{\nu^2} + \underbrace{\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2}}_{-\nu^2} = 0$$

(16'')

$$\frac{\partial^2 Q}{\partial \phi^2} + \nu^2 Q = 0$$

$$\frac{\rho^2}{R}\left[\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho}\right] + k^2\rho^2 = \nu^2$$

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

Radial Equation : $\cancel{x} = k\rho$ rescale

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$ (see next page) self-adjoint form

(Bessel function of order ν).

Solutions are the Bessel functions

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)}\left(\frac{x}{2}\right)^{2j}$$

A linear independent solution is (for $\nu \in \mathbb{Z}$)

$$Y_\nu(x) \equiv \lim_{p\to\nu} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)},$$

also $J_{-\nu} = (-1)^\nu J_\nu$ for $\nu \in \mathbb{Z}$

(Note: Since Φ is periodic in ϕ, then ν has to be integers)

$\Rightarrow$ Bessel Functions of integer Order

[If $\nu \notin \mathbb{Z}$, J_ν & $J_{-\nu}$ are linearly independent.]

$0 < x \ll \nu$:

$$\begin{cases} J_\nu(x) \sim \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{1}{2}x\right)^\nu & \nu \geq 0 \\[2ex] Y_0(x) \sim \dfrac{2}{\pi}\ln(x) & \\[2ex] Y_\nu(x) \sim -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{1}{2}x\right)^{-\nu} & \nu > 0 \end{cases}$$

check for self adjoint and compare with table

$$\underbrace{\frac{d}{dx}\left[x\frac{du}{dx}\right] - \frac{\nu^2}{x}u}_{\mathcal{L}u} + a^2 x u = 0$$ from table

$$\mathcal{L}u \quad + \lambda w(x) u = 0$$

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - \frac{\nu^2}{x}u + a^2 x u = 0$$

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)u = 0$$ $\underline{a = 1}$

↑

Bessel's Equation

For $x \gg \nu$, both ~~looks~~ like sine & cosine (18)

with decays $x^{-1/2}$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$$

For $x \simeq \nu$

$$J_\nu(\nu) \sim \frac{0.4473}{\nu^{1/3}}$$

$$Y_\nu(\nu) \sim -\frac{0.7748}{\nu^{1/3}}$$

Orthogonality condition : (With respect to b.c. $R(a)=0$)

For given ν, $\{J_\nu(x_{\nu n}\frac{\rho}{a})\}$ forms a complete set $\sim \{\sin(\frac{n\pi x}{L})\}$

☆ the order ν is dictated by the ϕ eq.

$$\int_0^a \rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})\, \delta_{n'n}$$

where $x_{\nu n}$ are roots of $J_\nu(x)$: $J_\nu(x_{\nu n}) = 0$,

graph : $J_\nu(x)$ has infinitely many roots !

~~sine/cosine~~

Recursion Relation: ~~For x < n~~

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

this is unstable (upward) see p.13

$$Y_{n+1}(x) = \frac{2n}{x} Y_n(x) - Y_{n-1}(x)$$

this is stable! (upward)

1. Need J_0, J_1, Y_0, Y_1, to start
2. Use recurrence to get others

   Y_n's are ok

   J_n's are unstable

Miller's Algorithm

If a recurrence relation is exponentially unstable in one direction, then the reverse iteration is stable! $[\, x_n = \frac{1}{2} x_{n+1} \bmod 1 \,]$

☆ The reverse iteration will converge to the desired seq. of fns times an unknown normalization factor (with arbitrary i.c.)

(Note: the recursion relation is invariant with respect to an overall factor.)

One possible normalization formula for Bessel fns (20)

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \quad (*)$$

<u>steps</u> : (Press)

1. Start with some arbitrary values as our seed for the downward iteration

$$K \gtrsim n + c\sqrt{n}$$

$c^2 \sim$ # of SF

$\sqrt{n}$ → convergent time

↑ order which we want

2. downward iterate to J_0 or J_1

~~3. Calculate J_0 or J_1 (later)~~

4. Normalized with (*) !

For $Y_\nu(x)$, upward is stable

steps: ① Calculate Y_0 or Y_1 (later)

② upward iterate to desired n !

Evaluation of J_0, J_1, Y_0, Y_1 :

(1) For $x < x_0 = 8$ (arbitrary small value of x),

- approx. $J_0(x)$ & $J_1(x)$ by rational functions in x.

$$\frac{P(x)}{Q(x)}$$

see NR

- approx. the 'regular' part of $Y_0(x)$ & $Y_1(x)$ by rational functions :

see NR

singular parts of $Y_{1,0}$ near 0. (from asymptotic)

$$Y_0(x) - \frac{2}{\pi} J_0(x) \ln(x)$$

$$Y_1(x) - \frac{2}{\pi}\left[J_1(x)\ln(x) - \frac{1}{x}\right]$$

---

(2) For $\underset{x_0}{8} < x < \infty$, use approx.

look like sine & cosine

$$J_{\substack{1,0\\n}}(x) = \sqrt{\frac{2}{\pi x}}\left[P_{\substack{n\\1,0}}\left(\frac{8}{x}\right)\cos(X_n) - Q_{\substack{n\\1,0}}\left(\frac{8}{x}\right)\sin(X_n)\right]$$

$$Y_{1,0}(x) = \sqrt{\frac{2}{\pi x}}\left[P_{\substack{n\\1,0}}\left(\frac{8}{x}\right)\sin(X_n) + Q_{\substack{n\\1,0}}\left(\frac{8}{x}\right)\cos(X_n)\right]$$

$$\underline{X}_n \equiv x - \frac{2n+1}{n}\pi$$

& $\underbrace{P_0, P_1}_{\text{even}}, \underbrace{Q_0, Q_1}_{\text{odd}}$ are polynomials.

→ Parameters are given in Press, Hart.

Spherical Bessel Functions

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R(r) + (kr)^2 R(r) = \ell(\ell+1)R(r)$$

from $\nabla^2\psi + k^2\psi = 0$

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+(1/2)}(x)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+(1/2)}(x)$$

Bessel functions of fractional order (later)

Error Function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \mathrm{erf}(x)$$

Note: the normalization is taken so that

$$\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt = 1$$

Rational Polynomial approximation

① $t = \frac{1}{1+px}$

$$\mathrm{erf}(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2}$$

$p, a_1, a_2, a_3 \rightarrow$ p. 147 $\quad |\epsilon| \leq 2.5 \times 10^{-5}$

$$\mathrm{erf}(x) = 1-(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5)e^{-x^2}$$

$$P', b_1, b_2, b_3, b_4, b_5 \rightarrow |\epsilon| \leq 1.5\times 10^{-7}$$

Gamma Functions

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

Integrating by parts,

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z}\, dt$$

$$= -e^{-t} t^{z}\Big|_0^\infty + z\int_0^\infty e^{-t} t^{z-1}\, dt$$

$$= z\Gamma(z)$$

For z integers:

$$\underline{\Gamma(n) = (n-1)!}$$

Polynomial approximations

p. 158

$$\Gamma(1+x) = 1 + \sum_{k=1}^{5,8} a_k x^k \qquad 0 < x < 1$$

$$|\varepsilon| \leq 5 \times 10^{-5}$$

$$|\varepsilon| \leq 3 \times 10^{-7}$$

use $\Gamma(z+1) = z\,\Gamma(z)$ to get other values for $z > 0$

Special values of $\Gamma(z)$

$z = 1 \Rightarrow 0! = 1$

$z = 0 \Rightarrow (-1)! = \infty$

$z = -n \Rightarrow \pm\infty$