$$J_0'(x) = -J_1(x)$$

and $$J_n'(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2}$$

} derivatives are known in terms of the functions

In this case, with two points $\{x_1, x_2\}$, we can use a 3rd order (cubic) polynomial.

$$P_3(x) = ax^3 + bx^2 + cx + d$$

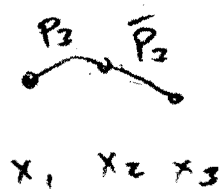
↑

four unknowns

$$P_3(x_1) = f(x_1) \qquad P_3(x_2) = f(x_2)$$

$$P_3'(x_1) = f'(x_1) \qquad P_3'(x_2) = f'(x_2)$$

} both $f(x)$ & $f'(x)$ are cont across intervals.



↑

four conditions

$\Downarrow$ algebra to solve for a, b, c, d

$$p(x)=\frac{\left(1-2\frac{(x-x_1)}{(x_1-x_2)}\right)(x-x_2)^2}{(x_1-x_2)^2}f(x_1)+\frac{\left(1-2\frac{(x-x_2)}{(x_2-x_1)}\right)(x-x_1)^2}{(x_1-x_2)^2}f(x_{\cancel{1}\,2})$$

$$+\frac{(x-x_1)(x-x_2)^2}{(x_1-x_2)^2}f'(x_1)+\frac{(x-x_2)(x-x_1)^2}{(x_1-x_2)^2}f'(x_2)$$

$$P(x) = \frac{(1-2(x-x_1))(x-x_2)^2}{(x_1-x_2)^2} f(x_1) + \frac{(1-2(x-x_2))(x-x_1)^2}{(x_1-x_2)^2} f(x_2) + \frac{(x-x_1)(x-x_2)^2}{(x_1-x_2)^2} f'(x_1) + \frac{(x-x_2)(x-x_1)^2}{(x_1-x_2)^2} f'(x_2)$$

Wrong in DeVries

next page

General form for more points

see DeVries p. 94-95.

Cubic Splines

As we have seen, if we have tabulated values of $f(x_i)$ & $f'(x_i)$, then Hermite interpolation

$\longrightarrow$ high order accurate approx.

☆ But, in most cases, $\{f'(x_i)\}$ are not available!

$\rightarrow$ Then, Lagrange interpolation gives continuous $P_N(x)$ but $P_N'(x)$ will not be.

Solution : To construct a cubic approximation with continuous derivatives.

- tradeoff: need three pts (two intervals) to define $P(x)$.

$$P(x) = a_j(x-x_j)^3 + b_j(x-x_j)^2 + c_j(x-x_j) + d_j$$

$$x_j < x \le x_{j+1}$$

steps:

① $P(x)$ to match data point at x_j & x_{j+1} (20)

$$P(x_j) = P_j = f(x_j) = d_j \longrightarrow d_j = f(x_j)$$

$$P(x_{j+1}) = P_{j+1} = f(x_{j+1}) = a_j h_j^3 + b_j h_j^2 + c_j h_j + P_j$$

$$\begin{bmatrix} P_j = P(x_j) \\ h_j = (x_{j+1} - x_j) \end{bmatrix} \downarrow$$

$$P_{j+1} = a_j h_j^3 + b_j h_j^2 + c_j h_j + P_j \quad (*)$$

Note: we know P_j, P_{j+1}
$f(x_j)$ $f(x_{j+1})$

unknown a_j, b_j, c_j

② Look at derivatives of $P(x)$:

$$P'(x) = 3a_j(x - x_j)^2 + 2b_j(x - x_j) + c_j$$

$$P''(x) = 6a_j(x - x_j) + 2b_j$$

3 Assume we have tabulated values of $\{f''(x_j)\}$.

then $P''(x_j) = P_j'' = 2b_j$

$$\Rightarrow \quad b_j = P_j''/2$$

$$P_{j+1}'' = 6a_j h_j + 2b_j$$

$$\Rightarrow \quad a_j = \frac{1}{6}\,\frac{P_{j+1}'' - P_j''}{h_j}$$

then from (*), we can solve for c_j

$$P_{j+1} - P_j = \left(\frac{1}{6}\,\frac{P_{j+1}'' - P_j''}{\cancel{h_j}}\right) h_j^{\cancel{3}\,2} + \left(\frac{P_j''}{2}\right) h_j^2 + c_j h_j$$

$$\frac{P_{j+1} - P_j}{h_j} = \frac{h_j P_{j+1}'' - h_j P_j''}{6} + \frac{3 h_j P_j''}{3 \cdot 2} + c_j$$

$$c_j = \frac{P_{j+1} - P_j}{h_j} - \frac{h_j P_{j+1}'' + 2 h_j P_j''}{6}$$

$\rightarrow$ All unknowns are in terms of P_j & P_j''

4 Now, we go back to solve for the $\{P_j''\}$ by requiring the continuity of $P(x)$ <u>across intervals</u>.

For $x \in [x_j, x_{j+1}]$:

$$P(x) = \frac{P_{j+1}'' - P_j''}{6h_j}(x-x_j)^3 + \frac{P_j''}{2}(x-x_j)^2 + \left[\frac{P_{j+1} - P_j}{h_j} - \frac{h_j P_{j+1}''}{6} - \frac{h_j P_j''}{3}\right](x-x_j) + P_j$$

$$P'(x) = \frac{P_{j+1}'' - P_j''}{2h_j}(x-x_j)^2 + P_j''(x-x_j) + \left[\frac{P_{j+1} - P_j}{h_j} - \frac{h_j P_{j+1}''}{6} - \frac{h_j P_j''}{3}\right]$$

For $x \in [x_{j-1}, x_j]$:

$$P'(x) = \frac{P_j'' - P_{j-1}''}{2h_{j-1}}(x-x_{j-1})^2 + P_{j-1}''(x-x_{j-1}) + \left[\frac{P_j - P_{j-1}}{h_{j-1}} - \frac{h_{j-1} P_j''}{6} - \frac{h_{j-1} P_{j-1}''}{3}\right]$$

Equating them at $x = x_j$, we have

$$\frac{P_{j+1} - P_j}{h_j} - \frac{h_j P_{j+1}''}{6} - \frac{h_j P_j''}{3} =$$

$$\frac{P_j'' - P_{j-1}''}{2\cancel{h_{j-1}}} h_{j-1}^{\cancel{2}} + P_{j-1}'' h_{j-1}$$

$$+ \left[\frac{P_j - P_{j-1}}{h_{j-1}} - \frac{h_{j-1} P_j''}{6} - \frac{h_{j-1} P_{j-1}''}{3} \right]$$

– Putting all terms with P_j'' on the left, we have,

$$\frac{3}{6} - \frac{6}{6} + \frac{2}{6} = -\frac{1}{6} \qquad \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$$

$$P_{j-1}'' h_{j-1} \left(\frac{1}{2} - 1 + \frac{1}{3} \right) - \frac{P_j'' h_j}{3} - P_j'' h_{j-1} \left(\frac{1}{2} - \frac{1}{6} \right)$$

$$- \frac{h_j P_{j+1}''}{6} = \frac{P_j - P_{j-1}}{h_{j-1}} - \frac{P_{j+1} - P_j}{h_j}$$

☆

$$\boxed{h_{j-1} P_{j-1}'' + (2h_j + 2h_{j-1}) P_j'' + h_j P_{j+1}'' = 6 \frac{P_{j+1} - P_j}{h_j} - 6 \frac{P_j - P_{j-1}}{h_{j-1}}}$$

Since this is done by linking adjcant intervals, we have only N-2 equations.

But, there will be $N \{P_j''\}$!

standard choices :

① natural spline : $P_1'' = P_N'' = 0$.

Then we have the following equation for P_j'' :

$$\begin{pmatrix} 1 & 0 & & & & \\ 0 & 2(h_1+h_2) & h_2 & & & \\ & h_2 & 2(h_2+h_3) & h_3 & & \\ & & & \ddots & & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & 0 \\ & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1'' \\ P_2'' \\ \vdots \\ P_N'' \end{pmatrix} = \begin{pmatrix} 0 \\ 6\dfrac{P_3-P_2}{h_2} - 6\dfrac{P_2-P_1}{h_1} \\ \vdots \\ 6\dfrac{P_N-P_{N-1}}{h_{N-1}} - 6\dfrac{P_{N-1}-P_{N-2}}{h_{N-2}} \\ 0 \end{pmatrix}$$

with these P_j'' solved,

$$\rightarrow P(x) = a_j (x - x_j)^3 + b_j (x - x_j)^2 + c_j (x - x_j) + d_j$$

Spline solution

in all intervals will be well defined!

- $P'(x)$, by construction, will be continuous across intervals.

(2) chose to have P_1' & P_N' to have specific values.

lead to another matrix eq'n on P. 98. DeVries

☆ these matrices are Tridiagonal.

∃ efficient way to invert them

→ later lecture on Linear Algebra.