In general, the histogram of \(\text{Integral}(N)\) for many different runs:

Note: The distribution of the average tends to be normal even if the distribution from which the average is computed is non-normal.

\[ \sigma \approx 1 \]

By Central Limit Theorem, for large \(N\),

dist \(\approx\) Gaussian with width \(\frac{1}{\sqrt{N}}\)

So, as \(N \to \infty\), estimate gets better \(\sim \frac{1}{\sqrt{N}}\)!

On the other hand, from your (ind. exercise),

\[ \Delta(N) (\text{SE}) \approx h^4 \text{ or } \frac{1}{N^4} \quad (h = \frac{(b-c)}{N}) \]

\[ (\varepsilon < 10^{-5} \text{, need } N \approx 20 \Rightarrow \frac{1}{N^4} \approx 10^{-5}) \]
the type of errors are different:

- **MC** - probabilistic
  - more pts $\rightarrow$ statistic gets better!
  - convergence depends on the averaging process $\frac{1}{N}$
    - AND not on dimensionality!
    - not on dimension of integral!

- **SR (others)** - approx. in fitting integrant
  - poly. better approx f(x) if grid is finer!

$N \rightarrow h^d$: for d-Dim.

$N(\frac{1}{2}h) = 2^d N(h)$

So, accuracy in SR (others) depends on dimension of integral!
bottom line:

- If $d < 4$, other method will in general requires less function calls to get good approx.

- If $d \geq 4$, MC is comparable in efficiency. For $d$ large, MC usually converge faster.

Practical way to test for accuracy

$$I_N - I_{N'} < \epsilon$$

$N \neq N'$ different set of $N$ #s.

If yes, $\overline{I}_N = \langle I_N, I_{N'} \rangle$

If no, $I_{2N}$ with different set of $2N$ #s

compare $I_{2N'}, \overline{I}_{2N}$
Example: thermodynamics (integrate over phase space)

\[ \tilde{u} = \frac{\int \text{e}^{-\epsilon/kT} \, dp^2w \, d\epsilon^2w}{\int \text{e}^{-\epsilon/kT} \, dp^2w \, d\epsilon^2w} \]

for \( N \) particles!

\[ \rightarrow \text{this type of high dim integral is} \]
basically intractable but with MC

(Answer might not be accurate but it serves well as rough estimate!)

Other advantages:

1. it is slow, but it always works!

2. if all else fails, MC might give estimate

3. singularities don't bother!
Multi-dimensional Numerical Integration

2D:

\[ I = \int \int_C f(x,y) \, dx \, dy \]

Difficulties with multi-D integrals:

1. \( N \) (# of grid pts) \( \propto h^{-d} \) 
   
   Evaluation pts 
   
   \[ \frac{N'}{N} = 2^1 \] 

2. Boundary Conditions

   1D \( \rightarrow \) pts (simple) 
   
   2D - curves 
   
   3D - surfaces 

arbitrary
Simple Rect Case

If \( a, b, c, d \) in \( \int_c^d \int_c^d f(x,y) \, dx \, dy \) are constant, the integrating domain is a rectangle.

\[
\begin{array}{c|cc}
 & c & d \\
\hline
a & | & |
\end{array}
\]

\[
\begin{array}{c|cc}
 & c & d \\
\hline
a & | & |
\end{array}
\]

Solution:

Nested 1D integrators:

1. Column first

\[
\int_c^b \int_c^d f(x,y) \, dx \, dy = \int_a^b G(x) \, dx
\]

where \( G(x) = \int_c^d f(x,y) \, dy \)

\( x \) is a parameter here!
Example: Trap rule

Main loop
Do
I = \frac{1}{2} G(x_0) + G(x_1) + \cdots + \frac{1}{2} G(x_N), Trap step
h = h/2
While (\% error < \pm 1)

Function G(x)

Do
I' = \frac{1}{2} f(x,y_0) + f(x,y_1) + \cdots + \frac{1}{2} f(x,y_N), inner trap step
h = h/2
While (\% error < \pm 1)

\# function G(x) of outer trap is another integrator.
\# Grid in x \& y are not necessarily idealized.
\# don't recalibrate inner G(x) if you need to halve the inner loop.

Row first:
\[
\int_a^b \int_c^d f(x,y) \, dx \, dy = \int_c^d F(y) \, dy
\]
where \( F(y) = \int_a^b f(x,y) \, dx \)
Other domains:

- Know boundary: \( y = f(x) \)

\[ I = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx \]

* Need function \( \sqrt{1-x^2} \) to describe boundary.

- Boundary without analytic function

- Conditional check for
  end pt at each column

- Monte Carlo
Another way to do this average!

- by uniformly but randomly sample pts in $[a,b]$.

Then $\langle f \rangle_N = \frac{1}{N} \sum_i f(x_i) \iff x_i \in P_{\text{uniform}}(x)$.

$$\Rightarrow \int_a^b f(x) \, dx \approx I_N = (b-a) \langle f \rangle_N$$

Monte Carlo estimate of integral.

2D $MC$ integration

$$I = \int_a^b \int_c^d f(x,y) \, dx \, dy$$

$\uparrow$ volume under surface

$$I \approx I_N = (b-a)(d-c) \frac{1}{N} \sum_i f(x_i,y_i)$$

$\uparrow$ area in domain

$\langle f \rangle_N$

- simple extension to higher $D$!
C) Improper Integrals

Types:

1. \( \int_a^b f(x) \, dx \), either \( a \) or \( b = \infty \).

2. \( \int_a^b f(x) \, dx \), \( f(x) \) is singular at \( x \in [a, b] \).

3. \( \int_a^b f(x) \, dx \), \( f(x) \to \infty \) at either \( a \) or \( b \) but \( \int_a^b f(x) \, dx < \infty \).

Solution:

3. - Newton-Cotes Open Formulas
   See, Press or Nakamura

2.3 - Monte Carlo Method

2 - Find singularity, partition \([a, b]\)
   \( \Rightarrow [a, x_i], [x_i, b] \), then use Newton-Cotes Open

2 - Choice of grid so that \( x_i \) won't land on \( x_c \)!
1. Remap range:

\[ \int_a^b f(x) \, dx = \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{t^2} f\left(\frac{1}{t}\right) \, dt \quad a, b > 0. \]

This can be used with either \( a \to \infty \) or \( b \to \infty \).

or other change of variables

⇒ see Press

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Monte Carlo Method

(effective for D ≥ 2 integrals)

(or \( f(x) \) similar in 2D, 3D)

grid pts \( \sim N^D \)

for \( D \) large, numerical costly!