Richardson Extrapolation

Recall, with two expressions of derivative $O(h)$ $\Delta f(\Delta f)$ and their associated errors.

we can combine them to form a new estimate with leading errors eliminated:

$$f(x) = f - \Delta f \ (O(h^2))$$

Richardson Extrapolation

- in a similar spirit is to achieve the same goal with a single expression –

* (the idea is general wherever the error of an expression is known)

Example with derivative
Central difference

1. With \( h \):
   \[ f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(x) + O(h^4) \]

   Using a half step size (but with a similar expression), we can get a better approximation to \( f'(x) \)!

   \( h \rightarrow \frac{h}{2} \), we can get a half approx to \( f'(x) \)!

2. With \( 2h \):
   \[ f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \frac{(2h)^2}{6} f'''(x) + O(h^4) \]

   By doubling the step size, the error is \( \frac{1}{4} \) \( (h^2) \)

With these two expressions, we can get a better approximation by eliminating the \( h^2 \) term!

\[ f'(x) - \frac{f'(x)}{4} = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} \]

\[ -\frac{h^2}{6} f''(x) + \frac{1}{4} \frac{(4h^2)}{6} f'''(x) + O(h^4) \]
\[ f'(x) = \frac{f(x-2h) - 2f(x-h) + 8f(x+h) - f(x+2h)}{12h} + O(h^4) \]

\[
(16 \times \frac{3}{12} = 2)
\]

\[ \Rightarrow \text{this expression for } f'(x) \text{ is accurate to } O(h^4) ! \]

Check with next result:

\[ D_3(h) = \frac{4D_1(h) - D_1(2h)}{3} \]

\[ = \frac{4(f(x^+)-f(x^-))}{2h} - \frac{(f(2h^+)-f(2h^-))}{4h} \]

\[ = \frac{f(2h^+)-8f(h^-)+8f(h^+)-f(2h^-)}{12h} \]
This procedure can be implemented numerically in a recursive way:

1. Let say $D_1(h)$ is our numerically computed result (diff, integration, etc.) with step size $h$.
   
   & we know its error to be of order $h^2$.

2. We can get a better estimate by calculating $D_1(2h)$ [the same expression with $2h$]

   Since $\Delta \approx h^2$, we know that $D_1(2h)$ will have $4 \times$ errors of $D_1(h)$.

(b) Calculate the error in $D_1(h)$ by:

\[ \Delta = D_1(2h) - D_1(h) \]

\[ \text{diff} = 3 \times \text{error in } D_1 \]

(c) $D_2(h) = D_1(h) + \Delta = D_1(h) - \frac{D_1(2h) - D_1(h)}{2^2 - 1}$

\[ \Rightarrow D_2(h) \text{ will be of } O(h^4) \]
(3) We can continue with the knowledge that $D_2(h)$ is of $O(h^4)$.

So, $D_2(2h)$ will have $2^4 \times$ the error of $D_2(h)$.

A better estimate will be:

$$D_2(h) = D_2(h) - \frac{D_2(2h) - D_2(h)}{2^4 - 1}$$

$$D_{i+1}(h) = D_i(h) - \frac{D_i(2h) - D_i(h)}{2^{2i} - 1}$$

Romberg Integration

- Since we know the error estimate for the trapezoidal rule to be $O(h^2)$, we can use Richardson extrapolation to improve our result.
Trapezoid rule:

\[
T_{m,0} \quad \text{(result for } N = 2^m N_0) \quad \quad h = \frac{(b-a)}{N}
\]

\[\uparrow \uparrow \quad \# \text{of } N \quad \text{level of} \quad \text{extrapolation} \quad \text{intervals halved!}\]

Then, \[
T_{m+1,0} = T_{m+1,0} - \left[ \frac{T_{m,0} - T_{m+1,0}}{2^2 - 1} \right]
\]

\[
= \frac{4T_{m+1,0} - T_{m,0}}{3}
\]

In general, we have the recursive scheme:

\[
T_{m+k, k} = \frac{4^k T_{m+k, k-1} - T_{m+k-1, k-1}}{4^k - 1}
\]
Romberg's Scheme

$T_{n,k}$

$m \downarrow$

$N = 2^m N_0$

$h = \frac{(b-a)}{N}$

$k$ (order of extrapolation)

Neville's Scheme

Going down this column is what you would do to improve accuracy without extrapolation.
Monte Carlo Integration

- Not based on approx. integrate with polynomials

\[ \int_a^b f(x) \, dx = \text{area under curve} \]

\[ \rightarrow I \sim \sum_i f(x_i) \cdot h = \overline{f} \cdot \frac{(b-a)}{N} \]

\[ = (b-a) \langle f \rangle \]

\[ \text{width \ average of } f \text{ on grid pts} \]

\[ \leftarrow \text{simplest approx. (rect rule)} \]
**Note:** with a fixed number of $N$.

**Case 1:**

\[ I_{mc}^1 = (b - a) \langle f \rangle \]

**Case 2:**

\[ I_{mc}^2 = (b - a) \langle f \rangle \]

\[ f \text{ is a uniform (constant)} \]

\[ \Rightarrow \text{In general, if } f(x) \text{ a constant in } [a,b], \]

the $I_{mc}$ will converge faster with less $N$!
Enhanced Monte Carlo

(Metropolis' Importance Sampling)

- let say we can find \( g(x) \) [later]

such that \( g(x) \propto f(x) \) in \([a,b]\)

and, we can evaluate the integral of \( g(x) \)

easily:

\[
y = h(y) = \int_a^x g(x') \, dx'
\]

then we have, \( \int_a^b f(x) \, dx \) = \( \int_a^b \frac{f(x)}{g(x)} g(x') \, dx \) change variable \( h(x) = y \).

\[
= \int_b^a \frac{h(y)}{h^{-1}(y)} \frac{f(h^{-1}(y))}{g(h^{-1}(y))} \, dy
\]

this will be nearly uniform.

* In terms of this new variable \( y = \int_a^x g(x) \, dx \), the integral will have a better convergence characteristics!
Note: To stop look at relative errors

\[ \frac{|I_{N'} - I_N|}{I_N} < \varepsilon \]

\( N', N \) different set "N" # of pts.

- MC integration is a random process!
- Each realization of a sequence of random # will give slightly different \( I_N \)

\[ \int_0^1 e^x \, dx \text{ with } N = 100, N = 400 \]

(C) distribution of 10,000 runs)

Fig. 4.12 in DeVries


(programs & graphs)