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$$\begin{cases} (b-a) = w_1 + w_2 \\ \frac{1}{2}(b^2 - a^2) = w_1 x_1 + w_2 x_2 \\ \frac{1}{3}(b^3 - a^3) = w_1 x_1^2 + w_2 x_2^2 \\ \frac{1}{4}(b^4 - a^4) = w_1 x_1^3 + w_2 x_2^3 \end{cases}$$

$[a, b] \to [-1, 1]$ simplify :

$$\Rightarrow \begin{cases} 2 = w_1 + w_2 \\ 0 = w_1 x_1 + w_2 x_2 \\ \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 \\ 0 = w_1 x_1^3 + w_2 x_2^3 \end{cases} \Rightarrow x_1^2 = x_2^2$$

$\Rightarrow$ $\underline{x_1 = -x_2}$ $\cancel{x_1 = x_2}$ ← assume distinct pts.

(b) $\Rightarrow$ $w_1 = w_2$

(a) $\Rightarrow$ $\underline{w_1 = 1}$

(c) $\Rightarrow$ $2x_1^2 = \frac{2}{3}$ $\underline{x_1 = 1/\sqrt{3} \quad x_2 = -1/\sqrt{3}}$

General Method to find $\{x_i\}$ & $\{w_i\}$

$$I_{[-1,1]} = \sum_{i=1}^{n} w_i f(x_i)$$

n-order Gaussian integration

Review on Orthogonal Polynomials

$\{\phi_n\}$ – a set of polynomials which satisfies the following orthogonality rule.

$$\int_a^b w(x)\, \phi_n(x)\, \phi_m(x)\, dx = \delta_{n,m}\, C_m$$

Mathematical way to define "sameness":

- If the two ϕ's are different $n \neq m$:

$$\int_a^b = 0$$

- If the two ϕ's are the same : $n = m$:

$$\frac{1}{C_m}\int_a^b = 1$$

normalization constant

☆ with respect to $\int_a^b$ & $w(x)$

Analogous to orthogonal basis in vector space :

regular Vector space		fn space
$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ $e_1 \quad e_2$	—	$\{\phi_i\}$
$\cdot$ (dot product)	—	$\int_a^b w(x) \quad dx$ ("dot" product in fn space)
$e_1 \cdot e_1 = 1$ $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$	—	$\frac{1}{c_m}\int_a^b w(x)\, \phi_n \phi_m \, dx = \delta_{n,m}$
	—	completeness

Example : Construction of orthogonal Legendre polynomials.

$\{\phi_i\} \rightarrow$ make up by $\{x^0, x^1, x^2, \ldots\} = \{u_m\}$

pick $\int_a^b w(x) \rightarrow \int_{-1}^{1} 1$ (☆ orthogonality depends on $\int_a^b$ & $w(x)$!)

(1) $\quad \phi_0 = c_0 u_0 = c_0$

$$\int_{-1}^{1} \phi_0 \phi_0 \, dx = c_0^2 \, x \Big|_{-1}^{1} = 2 c_0^2 \underbrace{=}_{\text{normalization}} 1$$

$$\Rightarrow \quad c_0 = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \boxed{\phi_0 = \frac{1}{\sqrt{2}}}$$

2 $$\phi_1(x) = \alpha_{11} u_1 + \alpha_{10} \phi_0$$

linear combination of what we have ϕ_0 & new element u_1

$$\int_{-1}^{1} \phi_0(x)\,\phi_1(x)\,dx = 0$$ ← orthogonality condition

$$\int_{-1}^{1} \phi_0 \left[\alpha_{11} u_1 + \alpha_{10}\phi_0\right] dx = 0$$

$$\underbrace{\int_{-1}^{1} \frac{\alpha_{11}}{\sqrt{2}}\, x\, dx}_{0} + \int_{-1}^{1} \alpha_{10}\frac{1}{2}\,dx = 0$$

odd fn.

$$\alpha_{10} = 0$$

$$\int_{-1}^{1} \phi_1(x)\,\phi_1(x)\,dx = 1$$ ← normalization

$$\int_{-1}^{1} \alpha_{11}^2 x^2\,dx = \alpha_{11}^2 \left.\frac{x^3}{3}\right|_{-1}^{1} = 1$$

$$\alpha_{11}^2\left(\frac{1}{3} + \frac{1}{3}\right) = 1$$

$$\alpha_{11} = \sqrt{\frac{3}{2}}$$

$$\Rightarrow \boxed{\phi_1(x) = \sqrt{\frac{3}{2}}\, x}$$

3 : $\phi_2(x) = \alpha_{22}\, u_2 + \alpha_{21}\phi_1(x) + \alpha_{20}\phi_0(x)$

$\vdots$ (see book) orthonormal conditions

$$\Rightarrow \boxed{\phi_2(x) = \sqrt{\frac{5}{2}}\,\frac{3x^2-1}{2}}$$

☆ So, how can we use these $\{\phi_n(x)\}$ to help use do Gaussian Integrals ?

Gaussian Integrals :

$$\int_a^b f(x)\, w(x)\, dx = \sum_{i=1}^{N} W_m f(x_i)$$

Note : this is more general than before, we had $w(x) = 1$. In general, we would like to know how to do classes of integrals with weights $w(x)$:

e.g. $\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx$ ← Gauss - Hermite

e^{-x^2} = $w(x)$

$w(x)=1$ $\int_{-1}^{1}$ Gauss-Legendre

Again, we have $\underline{2N}$ degrees of freedom :

$\{W_m\}$ & $\{x_m\}$

weights   abscissas

So, if $f(x)$ is a polynomial of order $2N-1$ or less,

then the Quadrature can be $\underline{\underline{\text{exact}}}$!

(recall $2N$ eqs with $2N$ unknowns, e.g. $N=2$)

To find $\{x_i\}$ & $\{w_i\}$, let $f(x) \in P_{2N-1}(x)$.

☆ Now, let choose a set of polynomials $\{\phi_N(x)\}$

which is orthogonal with respect to

the <u>integration limit</u> & <u>weight</u>.

$$\int_a^b w(x)\,\phi_n(x)\,\phi_m(x)\,dx = \delta_{n,m}\,C_m$$

<u>Note</u> : the orthogonal property depends on

$[a,b]$ & $w(x)$!!

- Now Divide $f(x)$ (a $(2N-1)$ polynomial) by $\phi_N(x)$ (a N order polynomial)

$$f(x) = \underbrace{q_{N-1}(x)}_{\text{quotient}} \overset{+ \text{ to } 2N-1}{\phi_N(x)} + \underbrace{r_{N-1}(x)}_{\text{reminder}}$$

$$\Rightarrow \int_a^b w(x)\, f(x)\, dx = \int_a^b q_{N-1}(x)\, \phi_N(x)\, w(x)\, dx + \int_a^b r_{N-1}(x)\, w(x)\, dx$$

Since $\{\phi_0, \phi_1, \dots, \phi_N\}$ are made up of polynomials from order 0 to N,

(completeness) $$q_{N-1}(x) = \sum_{i=0}^{N-1} \underbrace{q_i}_{\text{constants}} \phi_i(x)$$ $\leftarrow$ $q_{N-1}(x)$ can be written as a linear combination of $\phi_i(x)$

★ (completeness statement)

$$\Rightarrow \int_a^b q_{N-1}(x)\, \phi_N(x)\, w(x)\, dx = \sum_{i=0}^{N-1} q_i \int_a^b \phi_i(x)\, \phi_N(x)\, w(x)\, dx$$

$$= \sum_{i=0}^{N-1} q_i\, \delta_{i,N}\, C_N$$

$$= 0 \,! \quad , \quad \underline{\underline{i < N}}$$

— From oure hypothesis that

$$\int_a^b f(x)\, W(x)\, dx = \sum_{i=1}^{N} W_i\, f(x_i)$$

the Quadrature is exact for polynomial of order $\leq 2N-1$,

— this is also true:

$$\int_a^b \underbrace{\overset{f(x)}{q_{N-1}(x)\, \phi_N(x)}}_{\text{poly of } 2N-1} W(x)\, dx = \sum_{i=1}^{N} W_i\, q_{N-1}(x_i)\, \phi_N(x_i)$$

— If LHS $= 0$ from previous page for any abitrary $N-1$ order polynomial $q_{N-1}(x)$,

then

$$\underline{\underline{\phi_N(x_i) = 0}} \quad !$$

$$\boxed{\begin{array}{l}\text{☆☆ the abscissas of the quadrature must}\\ \text{be chosen to be the zeros of}\\ \qquad \phi_N(x) \; !\end{array}}$$

To find the weights:

Again from hypothesis, the quadrature need to be exact for polynomials of order $2N-1$,

$\curvearrowright$ (N-1) order Poly.

- choose $f(x) = l_{i,N}(x)$ ← lagrange interpolation polynomials

$(x-x_1)$ missing

e.g. $$l_{1,3} = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

recall that $$l_{i,N}(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\Rightarrow \int_a^b l_{i,N}(x)\, w(x)\, dx = \sum_{m=1}^{N} W_m\, l_{i,N}(x_m)$$

$$= W_i$$

☆☆ So, weights W_i can be obtained by the integrals $\int_a^b l_{i,N}(x)\, w(x)\, dx$ for a N order Gaussian Quadrature.

Example : For $N=2$ Gaussian-Legendre Quadrature

$$\int_{-1}^{1} \underset{\substack{\uparrow \\ W(x)=1}}{f(x)}\, dx = \sum_{i=1}^{N} W_i f(x_i)$$

→ For $\int_{-1}^{1} W(x)$, $\{\phi_N\}$ should be Legendre polynomials.

→ $N=2$, $\phi_2(x) = \sqrt{\frac{5}{2}}\,\frac{3x^2-1}{2}$

abscissas : zeros of $\phi_2(x)$:

$x_1 = -\sqrt{\frac{1}{3}}$, $x_2 = +\sqrt{\frac{1}{3}}$

→ Weights :

$$W_1 = \int_{-1}^{1} \ell_{1,2}\, dx = \int_{-1}^{1} \frac{x - x_2}{x_1 - x_2}\, dx$$

$$= \frac{1}{x_1 - x_2}\left[\frac{x^2}{2} - x_2 x\right]_{-1}^{1}$$

$$= \frac{1}{x_1 - x_2}\left[\cancel{\tfrac{1}{2}} - x_2 - \cancel{\tfrac{1}{2}} - x_2\right]$$

$$= \frac{-2x_2}{x_1 - x_2} = \frac{-2\sqrt{\frac{1}{3}}}{-2\sqrt{\frac{1}{3}}} = 1$$

$$W_2 = \int_{-1}^{1} l_{2,2}\,dx = \int_{-1}^{1} \frac{x-x_1}{x_2-x_1}\,dx$$

$$= \frac{1}{x_2-x_1}\left[\frac{x^2}{2} - x_1 x\right]_{-1}^{1}$$

$$= \frac{-2x_1}{x_2-x_1} = 1$$

Same results as before

→ Table on page 177