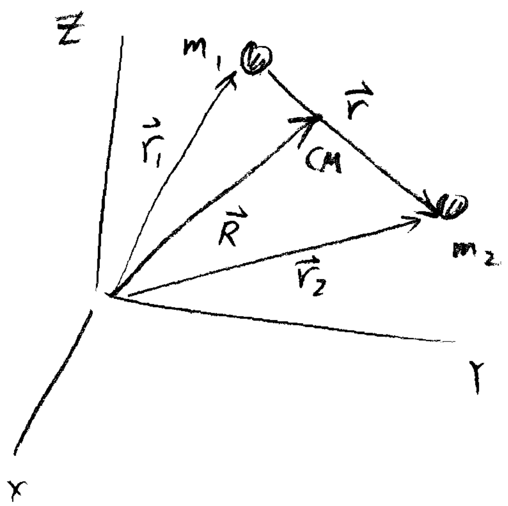
Central Force



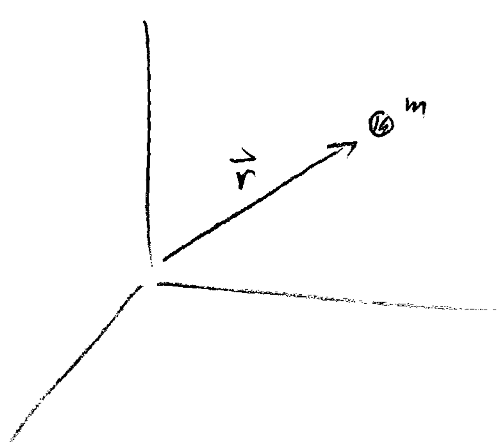
— central force :

interaction force is a function of $\vec{r}$ only !

☆ It can be shown that central force motion between (see ①′) m_1 & m_2 about their center of mass can always be reduced to an equivalent one body problem of a reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$ at a distance $\vec{r}$ form the fixed

center of force



$$\begin{cases} \text{mass } 1: & m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} \\ \text{mass } 2: & m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} \end{cases}$$

- Add these two equations:

$$m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = \vec{F}_{12} + \vec{F}_{21} \qquad (1)$$

Note: - the location of the center of mass is given by

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \& \quad \ddot{\vec{R}} = \frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1 + m_2}$$

- By Newton's 3rd laws we have $\vec{F}_{12} = -\vec{F}_{21}$
(force along $\vec{r} = \vec{r}_1 - \vec{r}_2$)

Then, Eq (1) becomes:

$$(m_1 + m_2)\ddot{\vec{R}} = \overset{\vec{F}_{12} - \vec{F}_{21}}{0} \quad \text{gives} \quad \ddot{\vec{R}} = 0$$

$\Rightarrow$ the center of mass is moving at constant velocity.

- Take ref. frame, in which the cm is at rest.

- Now, let consider the relative motion between m_1 & m_2, (1)

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2}$$

$$= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}, \quad (\vec{F}_{12} = -\vec{F}_{21}) \quad (2)$$

- With $m = \dfrac{m_1 m_2}{m_1 + m_2}$ (reduced mass),

(2) $\Rightarrow$ $\underline{m\ddot{\vec{r}} = \vec{F}_{12}(\vec{r})}$

- The two body problem can be reduced to an equivalent one body problem with the reduced mass m!

- In addition, we assume the force is conservative and it depends on $r = |\vec{r}|$ only, i.e.,

$$f(r) = -\frac{\partial U}{\partial r}$$

- Then, we have the following two properties:

  (1) force is conservative, E is conserved.

  (2) $U(r)$ depends on r only, soln will be spherically symmetric $\rightarrow$ $\vec{L}$ is conserved.

  $\rightarrow$ if $\vec{L}$ is conserved, $\vec{r}$ must always $\perp$ to $\vec{L}$!
  meaning that motions must be in a plane.

- From energy conservation, we have

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r)$$

$$\frac{d}{dt} E = 0 = \frac{d}{dt}\left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r)\right)$$

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{r}^2\right) = -\frac{d}{dt}\left(U + \frac{1}{2} m r^2 \dot{\theta}^2\right)$$

LHS:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$$

$$= \frac{1}{2}m(2\dot{r})\ddot{r}$$

$$= m\dot{r}\ddot{r}$$

Note:

$$\frac{mr^2\dot{\theta}^2}{2} = \frac{1}{2mr^2}\overbrace{m^2r^4\dot{\theta}^2}^{\to\ \ell^2}$$

$$= \frac{\ell^2}{2mr^2}$$

ℓ^2 - magnitude of angular momentum $\vec{L}$ (Note: ℓ is also conserved)

RHS:

$$-\frac{d}{dt}\left(U + \frac{1}{2}mr^2\dot{\theta}^2\right) = -\frac{d}{dr}(\quad)\frac{dr}{dt}$$

$$= -\left[\frac{d}{dr}\left(U(r) + \frac{\ell^2}{2mr^2}\right)\right]\dot{r}$$

$$= \left(f(r) - \frac{\ell^2}{mr^3}\right)\dot{r}$$

Putting them back together, we have,

$$m\ddot{r}\dot{r} = \left(-\frac{\ell^2}{mr^3} + f(r)\right)\dot{r}$$

$$m\ddot{r} = -\frac{\ell^2}{mr^3} + f(r)$$ <u>equation of motion</u>

(We will integrate this later. Now, we want to see what we can say by analyzing the $U'(r)$ term.)