3 Random Numbers Generation

A Uniformly Distributed Random #s.

- most common & easiest to implement (preinstalled functions in most softwares)
- "pseduo-random"
  - generated from a deterministic process → (very) weakly correlated but not random
  - reproducible with the same given seed

$$X_{n+1} = (aX_n + c) \bmod m$$

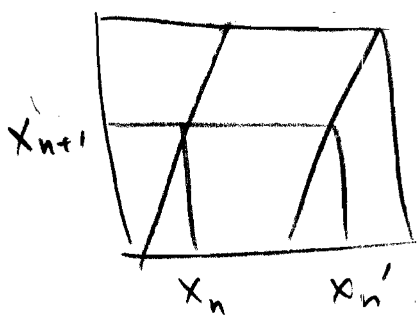
where a, c, m are large & properly chosen positive integers!

- Then for a given seed integer, the iterates will be "pseduo-random" # between $[0, m-1]$.
- To get $[0, 1]$ we can rescale $X_n / m-1$

Graphical view of pseudo-randomness :

(22)

$X_{n+1} = 2 X_n \mod 1$



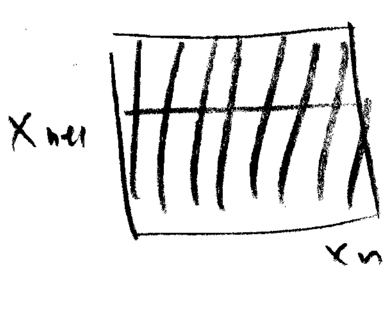
→ So X_{n+1} has come from two possible X_n's.

Further, for each X_n there are two possible X_{n-1}'s.

☆ In general, X_n has 2^n possible histories corresponds to 2^n possible X_0's!

☆ So, the correlation between X_n & X_0 decreases exponentially fast in n.

We can make our pseudo-generator better by choosing a as large as possible!



– X_{n+1} will have a (large) X_n's!

→ The optimal choice depends on the computer and its chip set!

[see Wong or Press for the common choices].

☆ Problem : → the system is still deterministic
→ pts don't fill out the full space,
pts stay on slices.

Improvement by shuffling

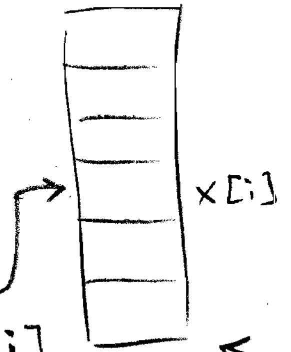
☆ As we mentioned, X_{n+1} is related to X_n by a function! (Seq correlation).

→ this can be broken up by shuffling.

③ refill x[i] by Y after output.

② get another ran # (minimal method as before)

Y use it to get random [i]



x[i] → output

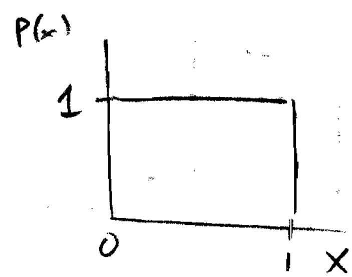
NTAB = 32 in ran1()

① array of pseudo-random # from before method

Random numbers with non-uniform distributions

- Most Random # generators give uniform distribution in $[0, 1]$

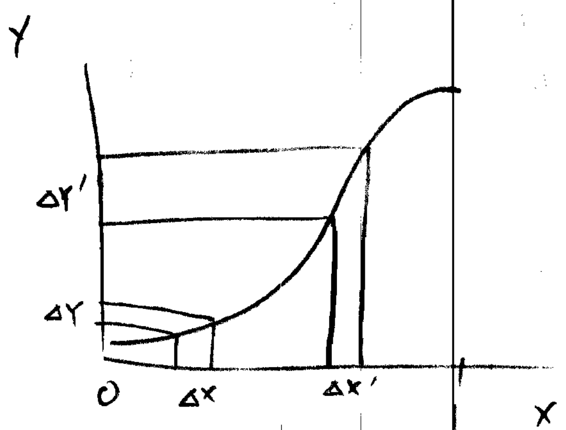
$$p(x)\,dx = 1\,dx \qquad 0 < x < 1$$
$$= 0 \qquad \text{otherwise}$$



- So, how can we generate random # with other distributions?

Method 1 : Tranformation Method

- Consider a function $Y = f(x)$ and we know that x is distribution uniformly in $[0, 1]$.



☆☆ Depending on the slope of $f(x)$, a group of points in Δx might scatter or concentrate in Y!
$\Delta Y'$ $\qquad$ ΔY

☆☆ And, from the conservation of probability, we know all pts in Δx will map to ΔY!

P - probability

i.e. $P(\Delta x) = P(\Delta Y)$

In terms of probability distribution function,

we have: $|p(x)\,dx| = |p(y)\,dy|$ (*)

→ Since Δy (dy) will scatter or concentrate due to $f(x)$,

$p(y)$ will NOT be uniform!

☆ We can use this fact to generate non-uniform distribution from uniform one.

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let see how this works ...

(*) → $$p(y) = \left|\frac{dx}{dy}\right| p(x).$$

example 1 : $y = -\ln x$ ($f(x)$)

derivative ↙ ↘ inverse

$$\frac{dy}{dx} = -\frac{1}{x} \qquad x = e^{-y}$$

derivative inverse

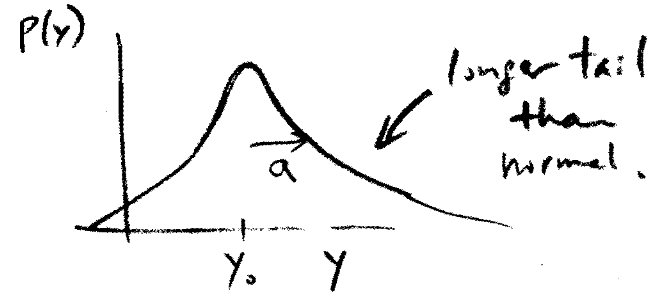
$$p(y) = \left|-\frac{1}{x}\right|^{-1} = e^{-y}$$

$= |x|$

So, $(p(y)\,dy = e^{-y}\,dy)$ - y is distributed exponentially!

example 2 : Lorentzian dist

$$P(y) = \left|\frac{dx}{dy}\right| = \frac{1}{\pi}\left[\frac{a}{(y-y_0)^2 + a^2}\right]$$



Integrating the above equation we have:

$$x = \frac{1}{\pi}\,\mathrm{atan}\left(\frac{(y-y_0)}{a}\right)$$

Find the inverse :

$$y = y_0 + a\tan(\pi x)$$

→ So, if we take a uniform dist x & put it thru this transformation, the transformed values y will be dist. according to Lorentzian.

(B) Random numbers with non-uniform distribution

(4')

Transformation method

General : $|\hat{\rho}(y)\, dy| = |\rho(x)\, dx|$

$$\hat{\rho}(y) = \rho(x) \left|\frac{dx}{dy}\right|$$

exponential : $y(x) = -\ln x$

$$x = e^{-y}$$

$$\hat{\rho}(y) = 1 \left|\frac{dx}{dy}\right| = e^{-y}$$

normal : $y_1 = \sqrt{-2\ln x_1}\, \cos 2\pi x_2$

$$y_2 = \sqrt{-2\ln x_1}\, \sin 2\pi x_2$$

$$x_1 = \exp\left[-\frac{1}{2}\left[y_1^2 + y_2^2\right]\right]$$

$$\Rightarrow \quad x_2 = \frac{1}{2\pi} \arctan \frac{y_2}{y_1}$$

$$\left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| = \left[\frac{1}{\sqrt{2\pi}} e^{-y_1^2/2}\right] \left[\frac{1}{\sqrt{2\pi}} e^{-y_2^2/2}\right]$$

$\rho(y_1, y_2) = \left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| \rho(x_1, x_2)$ = product of two indep. gaussians

<u>Note</u>: For the method to work, we need to have (1) $p(y) = \frac{dx}{dy}$ to be integrable

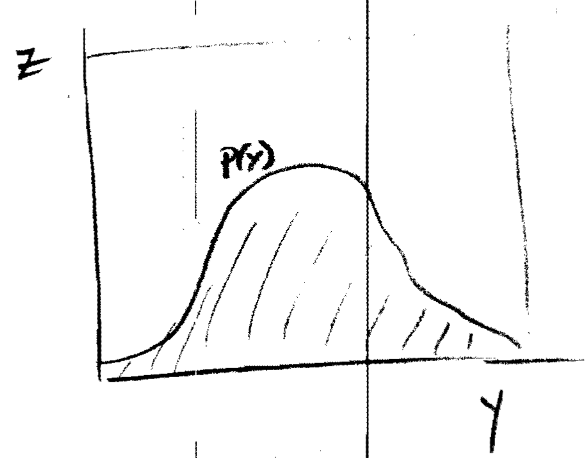
(2) $x = g(y)$ from (1) can be inverted to give $y = g^{-1}(x)$.

note: $p(x)$ is uniform.

<u>Method 2</u>: Rejection Method

- For an arbitrary $p(y)$!

Consider the graph of this function <u>geometrically</u>,



- let say we can choose a point in the z-y plane uniformly
- the total prob that we will choose a point (z, y) "according to $p(y)$" will be given by the ratio of $\frac{\text{Area (shaded)}}{\text{total area}}$!

procedure:

- Use two independent uniform random generators: (z, y).
- Now to decide if $(z, y) \in$ shaded area.

$\rightarrow$ $z \leq p(y)$ yes

$z > p(y)$ No.

- Taking only the "yes" pts, then the y values will be dist according to $p(y)$ - the distribution of y values!
  ("histogram")