Lecture 4 Fourier Analysis

I. Fourier Series

A. We have used Taylor's series extensively:

- allows to expand $f(x)$ near a point x^*
  as an infinite series of polynomials.

$$f(x) = f(x^*) + f'(x^*)(x-x^*) + \frac{f''(x^*)}{2!}(x-x^*)^2 + \ldots$$

drawback: local expansion around x^* is
not suited ~~fir~~ for x far from x^*.
e.g. global rep. of $f(x)$ (a periodic function)

- Need $f(x)$ to be C^∞.

B. An alternative to series expand an arbitrary
function $f(x)$ within a domain $[-L, L]$:
Fourier Series (Fourier 1768-1830)

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Advantages :

- Not relies on special pt x^*
- particular suited for period functions
  - rythmic behaviors are common in nature
- work with $f(x)$ which has finite # of non-smooth or discontinuous pts.

Dirichlet's Theorem :

If $f(x)$ is periodic of period 2π

and if for $-\pi < x < \pi$,

$f(x)$ has a finite # of Max & Mins

$f(x)$ has a finite # of discontinuities

$\int_{-\pi}^{\pi} f(x)\,dx < \infty$

then the Fourier Series converge to $f(x)$

at all pts where $f(x)$ is continuous,

and at jumps, FS converge to $\frac{1}{2}(f_+ + f_-)$.

☆ For $f(x) \in [-L, L]$, we can alway assume $f(x)$ to be copies of itself for x outside $[-L, L]$

→ so that $f(x)$ is periodic with $2L$.

Then. by rescaling $[-L, L]$ to $[-\pi, \pi]$, the above theorem applies!

Note : Obviously, since we are only interested in $f(x) \in [-L, L]$, (3)

☆ - FS expansion only makes sense within this domain.

(it says nothing about $f(x)$ outside $[-L, L]$) .

— true for other orthogonal expansion.

Modern Language

— $\{ \sin \frac{n\pi}{L} x , \cos \frac{n\pi}{L} x \}$ forms a complete set in the space of functions within the domain $[-L, L]$.

Thus, any function $f(x)$ can be written as a linear combination of them.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x .$$

The bases satisfy the orthogonality conditions:

$$\int_{-L}^{L} \sin\frac{m\pi}{L}x \, \sin\frac{n\pi}{L}x \, dx = \begin{cases} L\,\delta_{n,m} & m \neq 0 \\ 0 & m = 0 \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \, \cos\frac{n\pi}{L}x \, dx = \begin{cases} L\,\delta_{n,m} & m \neq 0 \\ 2L & m = n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{m\pi}{L}x \, \cos\frac{n\pi}{L}x \, dx = 0 \qquad \forall n, m.$$

Using these conditions, one can easily show that

$$\begin{cases} a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, & a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, & b_0 = 0 \end{cases}$$

Alternatively, we can use exponential functions

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$f(x) = \sum_{-\infty}^{\infty} g_n \, e^{i\frac{n\pi}{L}x}$$

$$g_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx$$

Note: Σ is from $-\infty$ to ∞ instead of $0 \to \infty$ previously.

Fourier Transform ...

FS as we have seen is useful in describing $f(x)$ in a limited regions.

FT is useful in describing function in the infinite domain.

- Recall we have

$$f(x) = \sum_{-\infty}^{\infty} g_n e^{\frac{in\pi}{L}x}$$

$$g_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi}{L}x}\, dx$$

- Now, define $k_n = \frac{n\pi}{L}$ and $\Delta k = \frac{\pi}{L}$

Then,

$$f(x) = \sum_{-\infty}^{\infty} g_n e^{in\Delta k x}$$

$$\& \quad g_n = \frac{\Delta k}{2\pi}\int_{-L}^{L} f(x)\, e^{-in\Delta k x} dx$$

We now define

$$g_n = \frac{\Delta k}{\sqrt{2\pi}}\, g(n\Delta k) \qquad \text{(rescale)}$$

so that

$$g(n\Delta k) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x)\, e^{-in\Delta k x}\, dx$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \Delta k\, g(n\Delta k)\, e^{in\Delta k x}$$

Now, they look almost symmetric!

Take $L \to \infty$, then $n\Delta k$ becomes a continuous variable k and

$$\boxed{\begin{aligned} \mathcal{F}^{-1}\quad f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk \\ \mathcal{F}\quad g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \end{aligned}}$$

Note: my convention is e^{-ikx} for forward
- NR has e^{ikx} for forward
- MatLab has e^{-ikx} for forward also.

$\mathcal{F}(f(x)) = g(k)$ Fourier Transform

$\mathcal{F}^{-1}(g(k)) = f(x)$ Inverse Transform

Note: the even split of $\frac{1}{\sqrt{2\pi}}$ is the convention for physicist (Quantum mechanics)

e.g. FT is used to transform a wavefunction from position rep. to momentum rep.

i.e. $\psi(x)$ is a wavefunction in position x

$\mathcal{F}(\psi(x)) = g(k)$ will be the corresponding wave function in momentum space P.

e.g. $\psi(x) = \delta(x-x_0)$ a particle at $x=x_0$

$$g(k) = \int_{-\infty}^{\infty} \delta(x-x_0)\, e^{-ikx}\, dx = e^{-ikx_0}$$

plane wave with momentum $h \cdot 2\pi k$.

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closure condition (express f(x) as inverse transform)

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k') e^{ik'x}\, dk'}_{f(x)} \Big] e^{-ikx}\, dx$$

$$g(k) = \int_{-\infty}^{\infty} \Big[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x}\, dx \Big] g(k')\, dk'$$

$$g(k) = \int_{-\infty}^{\infty} \delta(k'-k)\, g(k')\, dk'$$

Here,

$$\delta(k'-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k'-k)x}\, dx$$

is the Dirac delta function.

It has the property that

$$1 \quad \cancel{\delta(k'-k) = \begin{cases} 1 & \text{if } k'=k \\ 0 & \text{if } k'\neq k \end{cases}}$$

And, also

$$2 \quad \int_{-\infty}^{\infty} g(k)\, \delta(k'-k)\, dk = g(k')$$

Properties of Fourier Transform

1 FT is linear :

a). Given $f_1(x)$, $f_2(x)$

$$\mathcal{F}(f_1(x)+f_2(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f_1(x)+f_2(x))\, e^{-ikx}\, dx$$

$$= \mathcal{F}(f_1(x)) + \mathcal{F}(f_2(x))$$

b) $$\mathcal{F}(\alpha f(x)) = \int_{-\infty}^{\infty} \alpha f(x) e^{-ikx} dx$$
$$= \alpha \mathcal{F}(f(x))$$

2 Scaling relation :

If $g(k) = \mathcal{F}(f(x))$

then $$\mathcal{F}(f(\alpha x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\alpha x) e^{-ikx} dx$$

$(x' = \alpha x)$ $$= \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\int_{-\infty}^{\infty} f(x') e^{-ikx'/\alpha} dx'$$
$$= \frac{1}{\alpha} g\left(\frac{k}{\alpha}\right) \qquad \alpha > 0$$

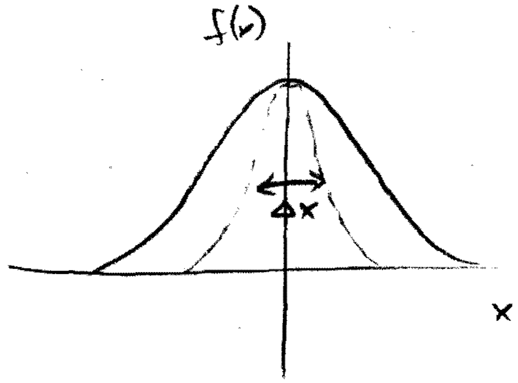
But, if $\alpha < 0$, $\mathcal{F}(f(\alpha x)) = -\frac{1}{\alpha} g\left(\frac{k}{\alpha}\right)$ because $\int_{-\infty}^{\infty} \rightarrow \int_{\infty}^{-\infty} = -\int_{-\infty}^{\infty}$
$x' = \alpha x \ (\alpha < 0)$

$$\boxed{\begin{aligned} \mathcal{F}(f(\alpha x)) &= \frac{1}{|\alpha|} g\left(\frac{k}{\alpha}\right) \\ \mathcal{F}^{-1}(g(\alpha k)) &= \frac{1}{|\alpha|} f\left(\frac{x}{\alpha}\right) \end{aligned}}$$

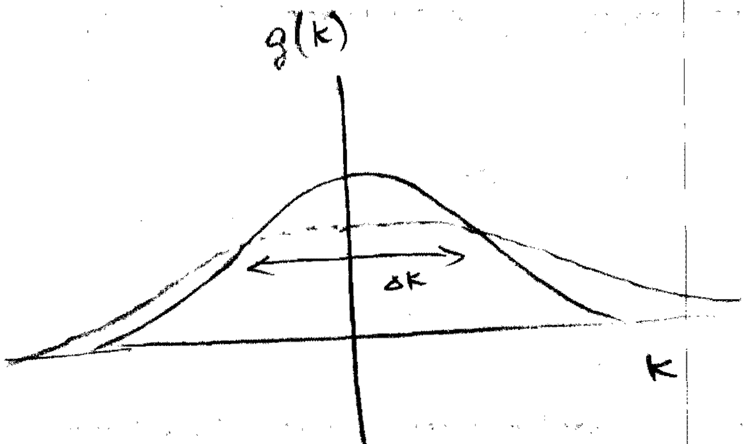
Insight in the Fourier Transform Pairs:

$f(x) \rightarrow f(\alpha x)$ (f gets narrower in x)

($\alpha < 1$)



$g(k) \rightarrow \frac{1}{|\alpha|} g\left(\frac{k}{\alpha}\right)$ (g gets wider in k)



☆ This is an intrinic property of the Fourier Transform and it has direct physical consequence!

$\Rightarrow$ Heisenberg's uncertainty Principle:

$\Delta x \Delta k \geq h$

(recall that wave fn in x, p are related by $\mathcal{F}$)

(C) Shift

$$\mathcal{F}(f(x-x_0)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x_0)\, e^{-ikx}\, dx$$

$$(x' = x - x_0) \qquad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-ik(x'+x_0)}\, dx'$$

$$= e^{-ikx_0}\,\mathcal{F}(f(x)) = e^{-ikx_0} g(k)$$

→ a shift in position x for f is the same as multipling a phase e^{-ikx_0} to $g(k)$!

Similarly, $\mathcal{F}^{-1}(g(k-k_0)) = e^{ik_0x} f(x)$

(D) Derivatives

$$g(k) = \mathcal{F}(f(x))$$

The $\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx$

Integrating by parts,

$$= \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)\Big|_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

assummed $f(x) \to 0$ as $x \to \pm\infty$

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$\mathcal{F}(f'(x)) = ik\, g(k)$

→ Derivatives in x $\approx$ multipling $g(k)$ by ik.

→ Important applicaton to solving DE.

★ change diff. equaton into algebric eq. in momentum space.

(E) Others

$f(x)$ real	$R\,g(k)$ even $Im\,g(k)$ odd
$f(x)$ Im	$R\,g(k)$ odd $Im\,g(k)$ even
$f(x)$ even	$g(k)$ even
$f(x)$ odd	$g(k)$ odd
$f(x)$ real & even	$g(k)$ real & even
$f(x)$ real & odd	$g(k)$ Im & odd
$f(x)$ Im & even	$g(k)$ Im & even
$f(x)$ Im & odd	$g(k)$ real & odd

(F) Parseval's Identity.

$$I = \int_{-\infty}^{\infty} f_1^*(x) f_2(x)\, dx$$ — total energy (t, ω) in EM — total prob in QM (x, k)

$$= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_1(k)\, e^{ikx}\, dk \right]^* \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_2(k') e^{ik'x} dk' \right] dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1^*(k)\, g_2(k') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x}\, dx \right] dk\, dk'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1^*(k)\, g_2(k')\ \delta(k'-k)\, dk\, dk'$$

$$= \int_{-\infty}^{\infty} g_1^*(k)\, g_2(k)\, dk$$

$|f(t)|^2$ - energy density at time t

($|f(x)|^2$ - prob density at x)

$|g(\omega)|^2$ - energy spectrum at ω

($|g(k)|^2$ - prob density at k)

Parseval's Identity $\Rightarrow$ total energy $= \int$ energy density dt

$= \int$ energy spectrum $d\omega$

Discrete Fourier Transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad (x, k) \text{ to } (t, \omega)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega$$

☆ they are contineous variables and experimental situation, data are sampled at discrete point and we don't have infinite range; we are actually in $[0, T)$. ☆ So, we can't really calculate the true FT !!

We need to consider DFT: → Similar to the derivation of FT, we start with Fourier series representation of data in the interval $0 \leq t < T$.

☆☆ If $T \gg P_{\text{period max}}$, the DFT will give good approx to FT.

Data: $f(m\Delta t)$, $m = 0, 1, \ldots, N-1$.

$$\begin{cases} f(t) = \displaystyle\sum_{-\infty}^{\infty} \tilde{g}_n\, e^{i 2\pi n t / T} & (**) \\ \tilde{g}_n = \dfrac{1}{T} \displaystyle\int_0^T f(t)\, e^{-i 2\pi n t / T}\, dt & (A) \end{cases}$$

☆ recall that this representation of $f(t)$ is periodic withe period T.

Similar in developing the FT, we introduce the conjugate variable

$$\Delta\omega = \frac{2\pi}{T}$$

and we define the DFT as rectangular rule of the integral $\int_0^T (*)$: (with $t = m\Delta t$)

$$g_n = g(n\Delta\omega) \equiv \sum_{m=0}^{N-1} f(m\Delta t)\, e^{-i2\pi n(m\Delta t)/T}$$

$$e^{-i2\pi n(m\Delta t)/T} = e^{-i2\pi nm/N} \quad \text{since } T = N\Delta t$$

(Note: $\underbrace{\frac{\Delta\omega}{2\pi} g_n}_{\text{DFT}} = \underbrace{\hat{g}_n}_{\text{coef. of FS}}$)

<u>Define</u>

$$\boxed{g_n = \sum_{m=0}^{N-1} f_m\, e^{-i2\pi mn/N}}$$

[Discrete Fourier Transform]

— We have only N data pts f_m, $0, \ldots, N-1$ & using FS rep., we assume that f_m are periodic with period N

☆ Thus, g_n will also be periodic with period N.

⇒ DFT maps N numbers $\{f_m\}$ into N numbers $\{g_n\}$.

But, for the inverse transform,

It is <u>not</u> obvious that simply truncating the sum in (**) is appropersiate!

meaning the inverse transfrom

$$f_m \stackrel{?}{=} \sum_{n=0}^{N-1} g_n \, e^{i 2\pi m n / N}$$ ← what we want!

— Below, we are going to show that this inversion is actually correct by utilitizing the idea that f_n can be orthogoronal with respect to a finite sum.

Consider the sum:

$$S_N = \sum_{k=0}^{N-1} e^{ik\alpha}$$

(1) If $\alpha = 0$, $\quad S_N = \sum_{0}^{N-1} 1 = N$

(2) If $\alpha \neq 0$,

$$\sum_{k=0}^{N-1} r^k = r^0 + r^1 + \ldots + r^{N-1} = \frac{1-r^N}{1-r}$$

$$r S_N = r + \ldots + r^N = \underbrace{1 + r + \ldots + r^{N-1}}_{S_N} + r^N - 1$$

$$(1-r) S_N = 1 - r^N$$

$$S_N = \frac{1-r^N}{1-r}$$

Apply to our case, we have

$$S_N = \sum_{k=0}^{N-1} e^{ik\alpha} = \frac{1-e^{i\alpha N}}{1-e^{i\alpha}}$$

To obtain an orthogonality condition, we need to have $S_n = 0$ for $\alpha \neq 0$.

$$\Rightarrow \quad e^{i\alpha N} = 1$$

(αN must be multiples of 2π!)

or $\quad \alpha = \frac{2\pi l}{N}$ with l an integer

$$\Rightarrow \quad S_N = \sum_{k=0}^{N-1} e^{i2\pi k l/N} = \begin{cases} N, & l=0 \ (\alpha=0) \\ 0, & l\neq 0 \ (\alpha\neq 0) \end{cases}$$

We now express l as the diff. between two integers n, m:

$$\boxed{\sum_{k=0}^{N-1} e^{i2\pi k(m-n)/N} = N\delta_{m,n}}$$

Consider the proposed inverser DFT, and substitute g_n with the DFT, we have:

$$\sum_{n=0}^{N-1} g_n e^{i2\pi mn/N} = \sum_{n=0}^{N-1}\Big[\overbrace{\sum_{k=0}^{N-1} f_k e^{-i2\pi kn/N}}^{g_n - DFT}\Big] e^{i2\pi mn/N}$$

$$= \sum_{k=0}^{N-1} f_k \underbrace{\sum_{n=0}^{N-1} e^{i2\pi n(m-k)/N}}$$

$$= \sum_{k=0}^{N-1} f_k \; N\delta_{m,k}$$

$$= N f_m$$

$e^{i2\pi nt/T} \to in(\frac{2\pi}{T})t$
$in(\Delta\omega)(m\Delta t)$
$inm\, 2\pi/N$
$i2\pi nm/N$

$\Delta\omega\Delta t = \frac{2\pi}{T}\frac{T}{N}$

Note: Compare to FS, you give up the eval of $f(t)$ for any t in $[0,T]$ so that the sum is now finite!

IDFT $\Rightarrow$ $\boxed{f_m = \frac{1}{N}\sum_{n=0}^{N-1} g_n e^{i2\pi mn/N}}$

Note: FS, FT, DFT are related but they are not identical!

e.g. DFT – use only finite sum ; IDFT – finite sum

FS – infinite sum ; g_ℓ – integral

FT – $\int_{-\infty}^{\infty}$; IFT – $\int_{-\infty}^{\infty}$

Fast Fourier Transform

$$\begin{cases} g_m = \sum_{n=0}^{N-1} f_n \, e^{-i2\pi mn/N} \\ f_n = \frac{1}{N} \sum_{m=0}^{N-1} g_m \, e^{i2\pi mn/N} \end{cases}$$

mapping N complex # into N complex #

- for each g_m, there are N operations (+ & *).
- for entire spectrum, there will be
  N^2 steps by direct application.

Spectral Analysis revolution

- FFT general became known in the mid-1960s.
  from the works of J Cooley & J. Tukey.
  - ☆ operation can be done in $N \log_2 N$ steps!
  - ☆ big saving in computation time.
  - ☆ became widely applied in eng. & physics.

– "clearest explanation" by Danielson & Lanczos 1942.

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Danielson & Lanczos Lemma

The N term sum in DFT can be rewritten as sum of two DFT each of length N/2.

For a particular n,

$$g_n = \sum_{m=0}^{N-1} e^{-2\pi i mn/N} f_m$$

$$= \sum_{\substack{m=0 \\ \text{even}}}^{N-1} f_m e^{-i2\pi mn/N} + \sum_{\substack{m=0 \\ \text{odd}}}^{N-1} f_m e^{-i2\pi mn/N}$$

$$= \sum_{j=0}^{N/2-1} f_{2j}\, e^{-i2\pi(2j)n/N} + \sum_{j=0}^{N/2-1} f_{2j+1}\, e^{-i2\pi(2j+1)n/N}$$

$$= \sum_{j=0}^{N/2-1} f_{2j}\, e^{-i2\pi jn/N/2} + e^{-i2\pi n/N} \sum_{j=0}^{N/2-1} f_{2j+1}\, e^{-i2\pi jn/N/2}$$

$$\boxed{\underbrace{g_n}_{\substack{\text{sum} \\ \text{over} \\ N}} = \underbrace{g_n^{even}}_{\substack{\text{sum} \\ \text{over} \\ N/2}} + e^{-i2\pi n/N}\, \underbrace{g_n^{odd}}_{\substack{\text{sum} \\ \text{over} \\ N/2}}}$$

← at first, this rewritting doesn't seem to save any steps. $N/2 + N/2 = N$

The Main point (Wong)

★ – **Not** all the g_n^e's g_n^o's are independent

– There is a strong symmetry among them

due to the periodic nature of $e^{-i2\pi jn/N/2}$

over n.

e.g. $N = 8$ (8 data pts)

Consider $n=0$

$$g_0 = g_0^{even} + e^{-i2\pi 0/N} g_0^{odd}$$

$$g_0^{even} \to \sum_{j=0}^{\frac{N}{2}-1} e^{-i2\pi j0/N/2} f_{2j} = \sum_{j=0}^{\frac{N}{2}-1} f_{2j}$$

$$g_0^{odd} \to \sum_{j=0}^{\frac{N}{2}-1} e^{-i2\pi j0/N/2} f_{2j+1} = \sum_{j=0}^{\frac{N}{2}-1} f_{2j+1}$$

★ $n=4$

$$g_4 = g_4^{even} + \left(e^{-i2\pi/N}\right)^4 g_4^{odd}$$

$$\left(\frac{n}{N/2} = \frac{4}{8/2} = 1\right) \to g_4^{even} \to \sum e^{-i2\pi j} f_{2j} = \sum f_{2j}$$

$$g_4^{odd} \to \sum e^{-i2\pi j} f_{2j+1} = \sum f_{2j+1}$$

So, $g_0^{even} = g_4^{even}$

$g_0^{odd} = g_4^{odd}$

Similarily, you show that

$$\left.\begin{aligned} g_0^{e(o)} &= g_4^{e(o)} \\ g_1^{e(o)} &= g_5^{e(o)} \\ g_2^{e(o)} &= g_6^{e(o)} \\ g_3^{e(o)} &= g_7^{e(o)} \end{aligned}\right\} \rightarrow$$

there are only 8 of them. (out of 16)

Applying D-L Lemma more steps:

$$g_n^{even} = \underbrace{g_n^{ee}}_{\text{even}} + \left(e^{-i2\pi/\frac{N}{2}}\right)^n \underbrace{g_n^{eo}}_{\text{odd}}$$

g_n^{ee} ↑ sum over N/4 g_n^{eo} ↑ sum over N/4

(8) → (16)

But, ☆ Again, by symmtry in the exp form, there are only 8 g_n^{ee} & g_n^{eo} !

$$g_n^{ee} = g_n^{eee} + (\underset{\downarrow}{\ \ })^n g_n^{eeo}$$

$g_n^{eee} \downarrow$ sum over N/8; $(\ )^n \to e^{-i2\pi/\frac{N}{4}}$

(contains only f_n N = 8 !)

☆ And, we have <u>8</u> of f_n's !

Note: Again, only 8 out of 16 $g_n^{eee(o)}$ are independent ! In fact, the 8 indep. $g_n^{eee(o)}$ are the 8 f_n's !

<u>steps counts in actual FFT :</u>

① Start with 8 f_n's (from bottom)

② form the 8 g_n^{ee}, g_n^{eo} from pairs of f_n^s (g_n^{eee}, g_n^{eeo}) (phase factor important)
(8 operations)

③ form the 8 g_n^{e}, g_n^{o} from pairs of g_n^{ee}, g_n^{eo}
(8 operations)

④ form the 8 g_n's from pairs of $g_n^e\ g_n^o$
(8 operations)

$$N \log_2 N$$

Total : $8 \times 3 = 8 \log_2 8 = 24$

$$N^2$$

NON-FTT : $8 \times 8 = 64$

Bottom line : for N large,

$$N^2 >> N \log_2 N \;!!$$

FTT is fast !

$(g_0^e + g_0^o)$

$g_0 = g^{10} + g^{11}$

$g_4 = g^{10} - g^{11}$

$g_1 = g^{12} - i\alpha g^{13}$

$g_5 = g^{12} + i\alpha\, g^{13}$

$g_2 = g^{14} - i\, g^{15}$

$g_6 = g^{14} + i\, g^{15}$

$g_3 = g^{16} - \alpha g^{17}$

$g_7 = g^{16} + \alpha g^{17}$

$\leftarrow$

$(g_0^e = g_0^{ee} + g_0^{eo})$

$g^{10} = g^{20} + g^{21}$

$g^{14} = g^{20} - g^{21}$

$g^{11} = g^{22} + g^{23}$

$g^{15} = g^{22} - g^{23}$

$g^{12} = g^{24} - i g^{25}$

$g^{16} = g^{24} + i g^{25}$

$g^{13} = g^{26} - i g^{27}$

$g^{17} = g^{26} + i g^{27}$

$\leftarrow$

$(g_0^{ee} = g_0^{eee} + g_0^{eeo})$

$g^{20} = f_0 + f_4$

$g^{21} = f_2 + f_6$

$g^{22} = f_1 + f_5$

$g^{23} = f_3 + f_7$

$g^{24} = f_0 - f_4$

$g^{25} = f_2 - f_6$

$g^{26} = f_1 - f_5$

$g^{27} = f_3 - f_7$

Sampling Theorem :

If $f(t)$ is continuous and sampled at Δ intervals in time and $f(t)$ is bandwidth limited so that

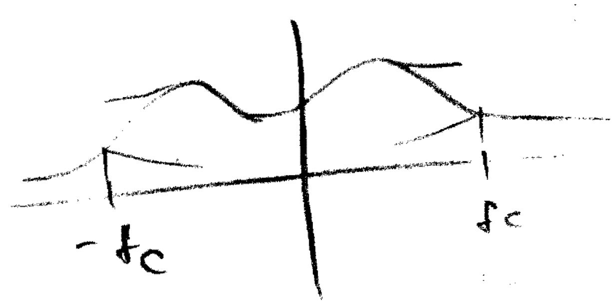
$g(f) = 0$ for all $|f| \geq f_c$, then

$f(t)$ is completely determined by f_n!

$$f(t) = \Delta \sum_{n=-\infty}^{+\infty} f_n \frac{\sin(2\pi f_c(t-n\Delta))}{\pi(t-n\Delta)}$$

– On the other hand, as our simple example shows, if $g(f) \neq 0$ for $|f| \geq f_c$, then spurious FT of high freq will move into the range $-f_c \leq f \leq f_c$.

This is called the aliasing problem.



☆

- can only be eliminated by low-pass filtering the original signal **before** **sampling**.

or - check $|g(f)|^2$ (power spectra) at f_c : If $|g(f)|^2 \gg 0$, then there is significant aliasing → need to be careful with DFT near f_c.

If $|g(f)|^2 \simeq 0$, then it is ok.

(☆ see back for data packing & freq range.)

(B) Freq. Leakage

- f_k rep. the whole discretized freq. bin.

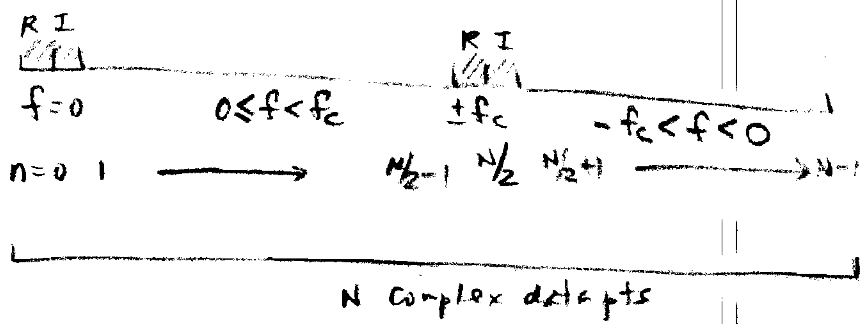
- $P(f_k) = |g(f_k)|^2$ will be some kind of average of $P(f)$ over the bin

The leakage of power to nearby bins (freq) is given by

(data packing)

H_n is periodic $\rightarrow$ $H_{-n} = H_{N-n}$

$$\left(H_n = \sum_{k=0}^{N-1} h_k \cdot e^{2\pi i k n/N} \right)$$

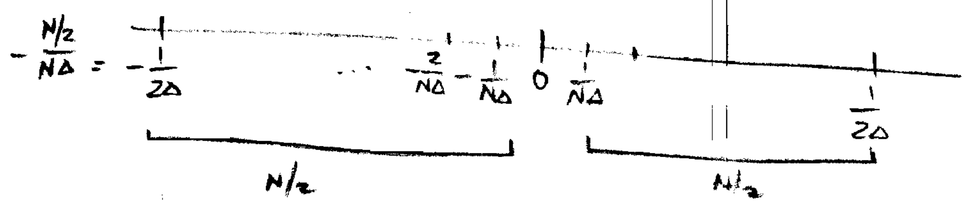


$$H_{-n} = e^{-2\pi i k n/N}$$

$$H_{N-n} = e^{2\pi i k (N-n)/N} = \cancel{e^{2\pi i k}}^{\,1} e^{-2\pi i k n/N}$$

(freq range)

★ This first



N+1

but $f(-\frac{1}{2\Delta}) = f(\frac{1}{2\Delta})$ so they are not independent!

$\rightarrow$ $\Delta f = \frac{1}{N\Delta} = \frac{1}{T}$

<u>Note</u>: Δ determines the max f $\longrightarrow$ $f_c = \frac{1}{2\Delta}$

T determines the Δf $\longrightarrow$ $\Delta f = \frac{1}{N\Delta} = \frac{1}{T}$

(13) freq. leakage

→ actual freq of data $\neq$ $m\Delta f$ evaluated by the transform.

e.g. $f(t) = \cos 3t$

Sample data every sec. $\Delta f = 1 Hz$

but natural freq $= \frac{2\pi}{3} Hz$

☆ FTT at $m\Delta f$, $g(m\Delta f)$ will try to estimate this with nearby freq.

— magnitude of transform leaks across freq bins.

Solution: ① increase resolution of freq.

$\Delta f = \frac{2\pi}{T}$ ← increase sampling time

Note: sampling rate Δt stays the same.

— just taking longer data set.

② try to match the sampling freq with actual freq.

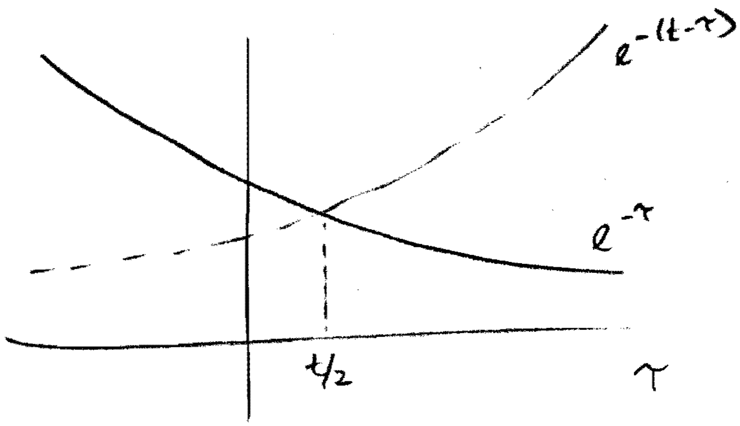
Convolution

(Mathematical)
Def: given two functions $p(t)$, $q(t)$,

$$(p \otimes q)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, q(t-\tau)\, d\tau$$

- sometimes refers to as the "Faltung"
  → german term for "folding".

example: $p(\tau) = q(\tau) = e^{-\tau}$
then $q(t-\tau)$ will look like:



Note: $q(t-\tau)$ is the mirror image of $q(\tau)$ with respect to $\tau = t/2$.

$q(t-\tau)$ can be generated by "folding" $q(\tau)$ over the vertical line $\tau = t/2$.

- Convolution arise naturally in many physical problems:

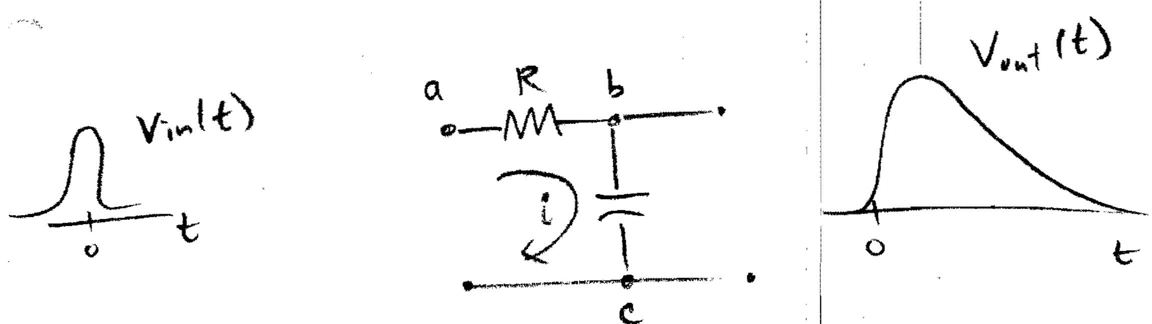
① : EM - solution to the Poisson's Eq.

$$\Phi(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d\underset{\sim}{r}'$$

$$\left(\nabla^2 \Phi = -\rho/\epsilon_0 \right)$$

Potential is the convolution of $\rho(\underset{\sim}{r}')$ (charge density) & $\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$ (distance weighting function).

② : to predict the linear response from a give input to a physical system.



$$i_R = \frac{V_a - V_b}{R}$$ (ressstor) current into pt b

$$i_c = C\frac{d}{dt}(V_b - V_c)$$ (capacitor) current out of pt b.

Continuity of current $\Rightarrow$

$$\frac{V_a - V_b}{R} = C\frac{d}{dt}(V_b - V_c)$$

Set reference $V_c = 0$.

$V_a = V_{in}$ & $V_b = V_{out}$

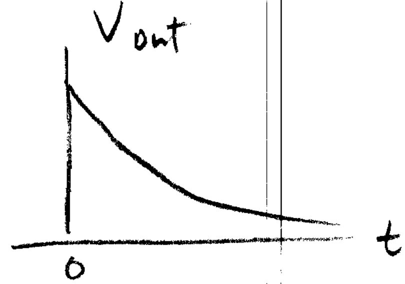
$$\frac{V_{in} - V_{out}}{R} = C\frac{d}{dt}V_{out}$$

$$\frac{d}{dt}V_{out} + \frac{V_{out}}{RC} = \frac{V_{in}}{RC}$$

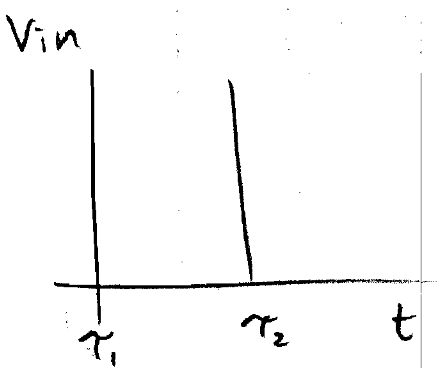
$$\Rightarrow \quad V_{out}(t) = e^{-t/RC}\left[\int_{-\infty}^{t} e^{\tau/RC}\, V_{in}(\tau)\, d\tau + C\right]$$

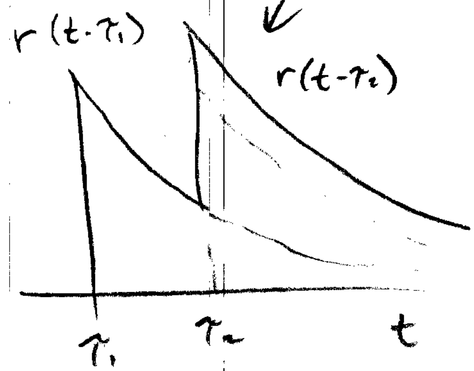
$(r(t))$

For $V_{in}(t) = \delta(t)$, $V_{out}(t) = \begin{cases} 0 & t<0 \\ \sim e^{-t/RC} & t \geq 0 \end{cases}$



— The system is linear (ODE is linear in V_{out}),
→ solutions to inputs simply add (superposition)





$V_{out}(t) = r(t-\tau_1) + r(t-\tau_2)$

$$V_{in}(t) = V_0\,\delta(t-\tau_1) + V_0\,\delta(t-\tau_2)$$

General continuous input: (sum → integral)

$$V_{in}(t) = \int_{-\infty}^{\infty} V_{in}(\tau)\,\delta(t-\tau)\,d\tau$$

Recall, $V_{in}(t) = \delta(t)$ (impulse), the system response is $r(t) = e^{-t/RC}$

So, $V_{out}(t) = \underline{\underline{\text{sum}}}$ of response to implulse inputs $V_{in}(\tau) \cdot \delta(t-\tau)$.

$$V_{out}(t) = \int_{-\infty}^{\infty} V_{in}(\tau)\, r(t-\tau)\, d\tau$$

$$= (V_{in} \otimes r)(t)$$

→ Convolution of $V_{in}(t)$ & $r(t)$ to delta input function!

Convolution Theorem (Fourier transform application)

$$\mathcal{F}(p \otimes q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [p \otimes q]\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, q(t-\tau)\, d\tau \right] e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(t-\tau)\, e^{-i\omega t}\, dt \right] d\tau$$

shifting property $\rightarrow$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t-\tau)\, e^{-i\omega t}\, dt = e^{-i\omega\tau}\, \underset{\substack{\uparrow \\ \mathcal{F}(g)}}{Q(\omega)}$$

Substitute back in, we have

$$\mathcal{F}(p\otimes g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(\tau)\, e^{-i\omega\tau} Q(\omega)\, d\tau$$

$$= \underset{\substack{\uparrow \\ \mathcal{F}(p)}}{P(\omega)}\, Q(\omega)$$

$$\Rightarrow \boxed{\mathcal{F}(p\otimes g) = \mathcal{F}(p)\,\mathcal{F}(g)}$$

---

usage to deconvolute :

Goal : know $V_{out}(t)$ and $r(t)$, need to find $V_{in}(t)$:

$$\mathcal{F}(V_{out}) = \mathcal{F}(V_{in})\,\mathcal{F}(r)$$

$$\mathcal{F}(V_{in}) = \frac{\mathcal{F}(V_{out})}{\mathcal{F}(r)} \Rightarrow V_{in}(t) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(V_{out})}{\mathcal{F}(r)}\right)$$

Correlation

- test for similarity between two signals!

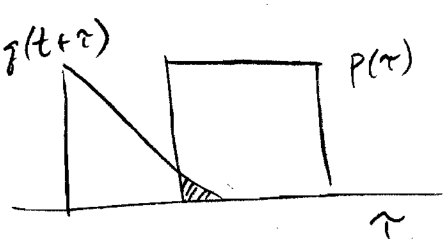
Def: $$p \odot g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau)\, g(t+\tau)\, d\tau$$

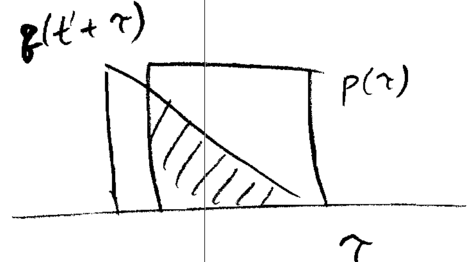
If p, g are real fns, then

$$p \odot g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, g(t+\tau)\, d\tau$$

---

test for similarity by <u>summing up overlaps</u>





→ $p(t)$ & $g(t)$ are <u>more correlated</u> at t' than t!
"doing the same thing"

– Two typical functions will be uncorrelated if

$p \otimes g = \langle p \rangle \langle g \rangle$ ← simply the product of p & g over the interval.

(if both $\langle p \rangle = \langle g \rangle = 0$, then $p \otimes g = 0$ for uncorrelated p & g)

Correlation and Fourier transform

$$\mathcal{F}[p \otimes g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [p \otimes g]\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau)\, g(t+\tau)\, d\tau \right] e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t+\tau)\, e^{-i\omega t}\, dt \right] d\tau$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau)\, e^{+i\omega\tau}\, Q(\omega)\, d\tau$$

$$= Q(\omega) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, e^{-i\omega\tau}\, d\tau \right]^*$$

$$= Q(\omega)\, P^*(\omega)$$

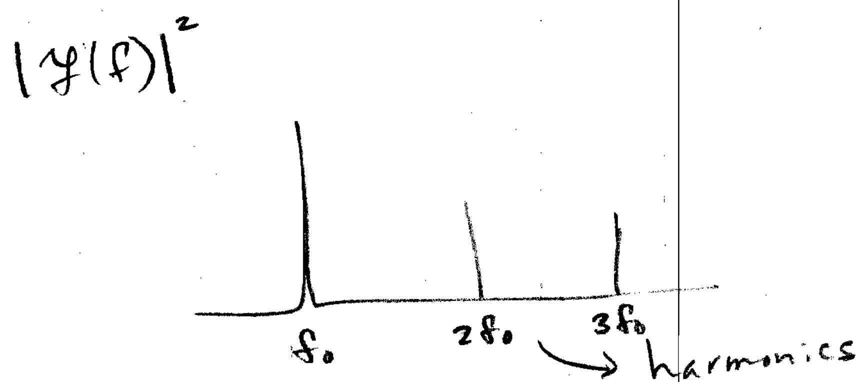
$$\boxed{\mathcal{F}[p \otimes g] = \mathcal{F}^*(p)\, \mathcal{F}(g)}$$

$$\mathcal{F}(p^2) = \mathcal{F}^*(p)\, \mathcal{F}(p) = |P(\omega)|^2$$

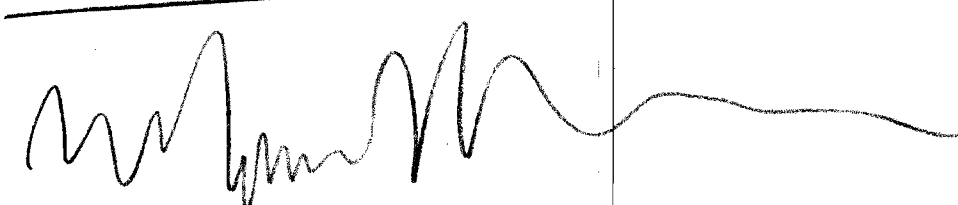
↑ auto correlation fun ↑ power spectrum.

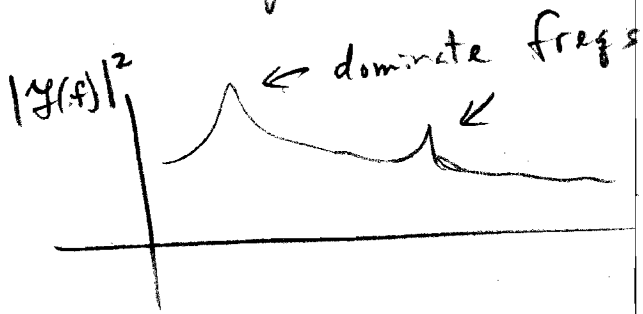
Spectral Analysis

- study of dynamics at different time scale. (or length)
  - periodic signal will have a pure frequency



  - General signal will not be pure tone



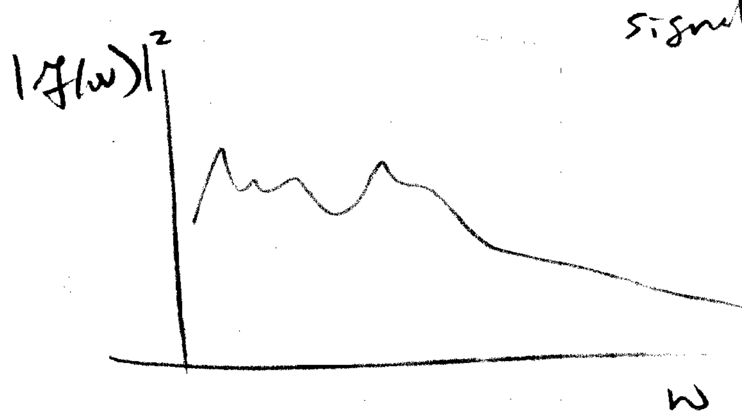


← contains many many frequencies.

– Spectral analysis is to look at signals in its freq. domain.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\omega)\, e^{i\omega t}\, d\omega$$

signal characteristic at a particular freq.



$$|P(\omega)|^2 = \mathcal{F}^*(\omega)\, \mathcal{F}(\omega) \equiv \text{power spectrum}$$ (total energy in a signal at ω)

<u>usage</u>

① decomposition of signal into its freq. components

② Noise or other type of filtering

- high pass
- low pass
- band pass

(If signal of interests is in a particular freq range)

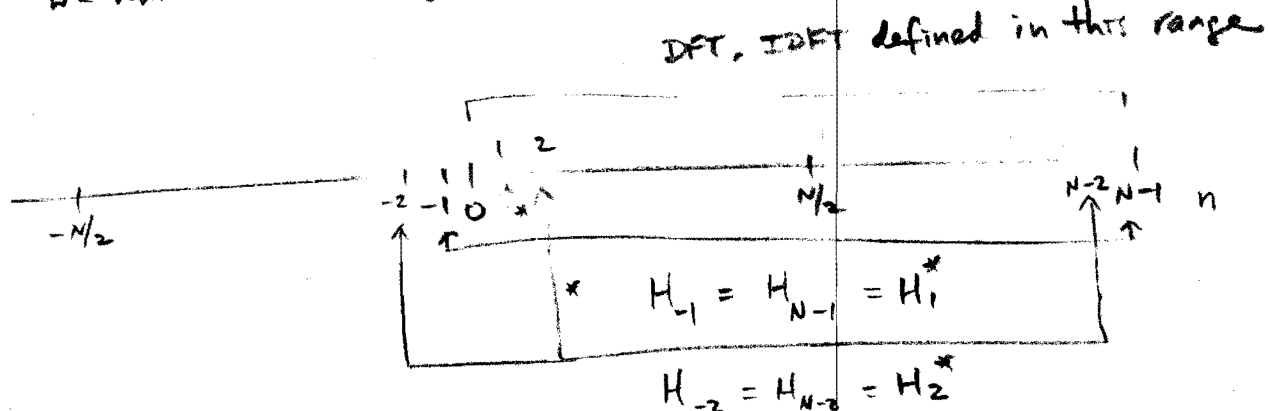
DFT: $$H_n = \sum_{k=0}^{N-1} h_k \, e^{-2\pi i k n/N}$$

IDFT: $$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n \, e^{+2\pi i k n/N} \quad (*)$$

For real data: $H_n = H_{-n}^*$

DFT is periodic with period N: $H_{-n} = H_{N-n}$

So we have the following symmetry in H_n's:



Note: $H_{-N/2} = H_{N/2}$

---

let write the IDFT in terms of cos & sin:

Separate the sum in $(*)$ into two parts:

$$h_k = \frac{1}{N}\left[\sum_{n=1}^{N/2-1} H_n e^{+2\pi i k n/N} + \sum_{n=N/2+1}^{N-1} H_n e^{+2\pi i k n/N} + H_0 e^{+2\pi i k 0/N} + H_{N/2} e^{+i\pi k} \right]$$

zero f

both of these have $N/2-1$ terms

critical f

The second term can be rewritten as follows using the symmetry :

$$H_n = H_{N-n}^* = H_{-n}^*$$

so, $\sum_{n=N/2+1}^{N-1} H_n \, e^{+2\pi i k n/N}$

$$= \sum_{n=N/2+1}^{N-1} H_{N-n}^* \, e^{+2\pi i k n/N}$$

$$= \sum_{n=1}^{N/2-1} H_n^* \, e^{+2\pi i k (N-n)/N} \qquad n \to N-n$$

$$= \sum_{n=1}^{N/2-1} H_n^* \, e^{-2\pi i k n/N} \qquad \cancel{e^{-2\pi i k}}^{\,1}$$

let $H_n = P_n + i g_n$, where P_n , g_n are real.

Then, we have

$$h_k = \frac{1}{N}\Big[\sum_{n=1}^{N/2-1} (P_n + i g_n)(\cos(2\pi k n/N) + i \sin(2\pi k n/N))$$

$$+ \sum_{n=1}^{N/2-1} (P_n - i g_n)(\cos(2\pi k n/N) - i \sin(2\pi k n/N))$$

$$+ P_0 + P_{N/2} \cos \pi k \Big]$$

$$= \frac{1}{N}\Big[\sum_{n=1}^{N/2-1}\Big\{ P_n \cos + \cancel{i P_n \sin} + \cancel{i g_n \cos} - g_n \sin$$

$$+ P_n \cos - \cancel{i P_n \sin} - \cancel{i g_n \cos} - g_n \sin \Big\}$$

$$+ P_0 + P_{N/2} \cos \pi k \Big]$$

$$h_k = \frac{1}{N}\Big[\sum_{n=1}^{N/2-1}\Big\{ \overbrace{2P_n}^{a_n} \cos(2\pi kn/N) - \overbrace{2g_n}^{b_n} \sin(2\pi kn/N)\Big\}$$

$$+ P_0 + P_{N/2} \cos \pi k \Big]$$

So, $\left.\begin{array}{l} a_n = 2\,\mathrm{Re}\,H_n \\ b_n = -2\,\mathrm{Im}\,H_n \end{array}\right\}$ for $f \neq 0, f_c$

Note: if we use NR convention with e^{ikx} in the forward direction, then $b_n = 2\,\mathrm{Im}\,H_n$.

$a_0 = \mathrm{Re}\,H_0 = P_0$

$b_0 = 0$

$a_{N/2} = \mathrm{Re}\,H_{N/2} = P_{N/2}$

$b_{N/2} = 0$

Note: $$P_0 = H_0 = \sum_{k=0}^{N-1} h_k \, \cancel{e^{2\pi i k 0/n}}^{\,1}$$

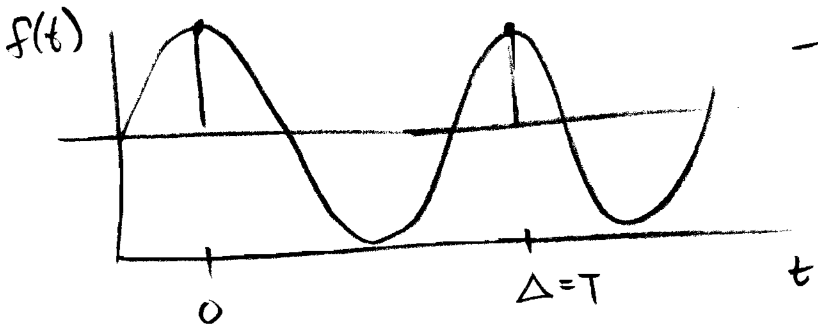
$$= \sum_{k=0}^{N-1} h_k$$

Note: this is $\varnothing$ off-set; in C arrays are 1 off-set.

Problems with DFT :

① Aliasing :

(undersampling)



— sampled every period $\Delta = T$,

$f_0, f_T, f_{2T}, \ldots$ will not look like a sine wave but a flat line !

⇒ For discretely sampled data, the highest freq which can be sampled is call the Nyquist freq.

$$f_c = \frac{1}{2\Delta}, \quad \Delta \text{ - sampling interval.}$$

In order words, if the highest freq in $f(t)$ is f_{max}, then in order to have good representation in DFT,

$$f_{max} \leq f_c = \frac{1}{2\Delta} \Rightarrow \Delta \leq \frac{1}{2 f_{max}}$$

samp. interval $\leq \frac{1}{2}$ shortest period.