$$H\varphi_n = E_n\varphi_n \qquad H = H^\circ + V'(x)$$

This perturbed Hamiltonian defines a new set of eigenenergies $\{E_n\}$ and eigenfunctions $\{\varphi_n\}$.

To first order approximation, assume

$$E_n \simeq E_n^0 + E_n^1$$

$$\varphi_n \simeq \varphi_n^0 + \sum_m a_{nm}^1 \varphi_n^0 \;,$$

$\{E_n^0\}$ eigenenergies & $\{\varphi_n^0\}$ eigfunctions of unperturbed H°!

Applying these approx to the time-independent S' equation,

$$(H^\circ + V')\left(\varphi_n^0 + \sum_m a_{nm}^1 \varphi_m^0\right) = (E_n^0 + E_n^1)\left(\varphi_n^0 + \sum_m a_{nm}^1 \varphi_m^0\right)$$

$$\left[H \qquad \varphi_n \qquad = \qquad E_n \qquad \varphi_n\right]$$

$$\cancel{H^\circ\varphi_n^0} + H^\circ \sum_m a_{nm}^1 \varphi_m^0 + V'\varphi_n^0 + V'\sum_m a_{nm}^1 \varphi_m^0 = \cancel{E_n^0\varphi_n^0} + E_n^0 \sum_m a_{nm}^1 \varphi_m^0 + E_n^1\varphi_n^0 + E_n^1 \sum_m a_{nm}^1 \varphi_m^0$$

(recall $H^\circ\varphi_n^0 = E_n^0\varphi_n^0$)

Now, keep only terms up to 1st order in perturbations, we have

$$H^0 \sum_m a'_{nm} \varphi_m^0 + V' \varphi_n^0 = E_n' \varphi_n^0 + E_n^0 \sum_m a'_{nm} \varphi_m^0$$

$$\boxed{\sum_m a'_{nm} E_m^0 \varphi_m^0 + V' \varphi_n^0 = E_n' \varphi_n^0 + E_n^0 \sum_m a'_{nm} \varphi_m^0}$$

Integrate both sides of eq. with $\int dx\, \varphi_n^0(x) \equiv \langle \varphi_n^0 |$,

we have,

$$\sum_m a'_{nm} E_m^0 \langle \varphi_n^0 | \varphi_m^0 \rangle + \langle \varphi_n^0 | V' | \varphi_n^0 \rangle =$$
$$E_n' \langle \varphi_n^0 | \varphi_n^0 \rangle + E_n^0 \sum_m a'_{nm} \langle \varphi_n^0 | \varphi_m^0 \rangle$$

By orthogonality, $\sum_m a'_{nm} E_m^0 \langle \varphi_n^0 | \varphi_m^0 \rangle = a'_{nn} E_n^0 \langle \varphi_n^0 | \varphi_n^0 \rangle$

so we have only two terms left,

So, finally, we have

$$\langle \varphi_n^0 | V' | \varphi_n^0 \rangle = E_n' \langle \varphi_n^0 | \varphi_n^0 \rangle$$

$$\boxed{E_n' = \frac{\langle \varphi_n^0 | V' | \varphi_n^0 \rangle}{\langle \varphi_n^0 | \varphi_n^0 \rangle}}$$