(1) Gauss-Hermite Quadrature

a). $$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \cong \sum_{k=1}^{N} w_k f(x_k)$$

For $N=2$, if $f(x) = P_{2N-1}(x)$ ($f(x)$ is a polynomial of order $(2N-1)$, then the equality can be made exact)

The necessary w_1, w_2 & x_1, x_2 can be solved from the following set of equations:

$$\begin{cases} f(x)=1: & \int_{-\infty}^{\infty} e^{-x^2}\, dx = w_1 + w_2 \\ f(x)=x: & \int_{-\infty}^{\infty} x e^{-x^2}\, dx = w_1 x_1 + w_2 x_2 \\ f(x)=x^2: & \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = w_1 x_1^2 + w_2 x_2^2 \\ f(x)=x^3: & \int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx = w_1 x_1^3 + w_2 x_2^3 \end{cases}$$

$$\int_{-\infty}^{\infty} x e^{-x^2}\, dx = \int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx = 0$$ since the integrands are odd!

$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$ can be integrated by considering

$$I^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx \int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)}\, dx\, dy$$

Now, consider this as an integral over the plane and we wish to change coordinates to (ρ, θ),

$$I^2(\alpha) = \int_0^{2\pi} \int_0^{\infty} e^{-\alpha \rho^2} \rho \, d\rho \, d\theta$$

$$= \frac{1}{2\alpha} \int_0^{2\pi} d\theta = \frac{\pi}{\alpha}$$

$$\Rightarrow \quad I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

Also, notice that $\quad -\frac{\partial}{\partial \alpha} I(\alpha) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} e^{-\alpha x^2} dx$

$$\frac{1}{2}\sqrt{\pi}\, \alpha^{-3/2} = \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx$$

So, for $\alpha = 1$, we have

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 e^{-x^x} dx = \frac{\sqrt{\pi}}{2}$$

This gives:

$$\begin{cases} \sqrt{\pi} = W_1 + W_2 & (1) \\ 0 = W_1 X_1 + W_2 X_2 & (2) \\ \dfrac{\sqrt{\pi}}{2} = W_1 X_1^2 + W_2 X_2^2 & (3) \\ 0 = W_1 X_1^3 + W_2 X_2^3 & (4) \end{cases}$$

Solving for W_1 from ② & substitute it into ④ gives:

$$-\frac{W_2 X_2}{X_1} X_1^3 + W_2 X_2^3 = 0$$

$$X_1^2 - X_2^2 = 0$$

Since, we want distinct roots,

$$\boxed{X_1 = -X_2}$$

Substitute this back into ②, we have

$$\boxed{W_1 = W_2}$$

Then, ① gives $\boxed{W_1 = \sqrt{\pi}/2}$

Then, ③ gives $\frac{\sqrt{\pi}}{2} = \sqrt{\pi} X_1^2$

$$\boxed{X_1 = 1/\sqrt{2}}$$

So, $\boxed{W_1 = W_2 = \sqrt{\pi}/2}$ and $\boxed{X_1 = X_2 = \pm 1/\sqrt{2}}$

↑

Note: they are the roots to $H_2(p) = 4p^2 - 2$ also!

b). $$\int_{-\infty}^{\infty} e^{-x^2} \cos x \, dx \cong \sum_{k=1}^{N} w_k f(x_k)$$

For $N=2$: $$\int_{-\infty}^{\infty} e^{-x^2} \cos x \, dx \simeq \frac{\sqrt{\pi}}{2} \cos\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{\pi}}{2} \cos\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \underline{1.347498}$$

For $N=5$: $$\int_{-\infty}^{\infty} e^{-x^2} \cos x \, dx \simeq W_0 \cos(x_0) + W_1 \cos(x_1) + W_1 \cos(-x_1) + W_2 \cos(x_2) + W_2 \cos(-x_2)$$

$$= \underline{1.380390}$$

$x_0 = 0.0$

$\pm x_1 = \pm 0.95857246$

$\pm x_2 = \pm 2.02018287$

$W_0 = 0.94530872$

$W_1 = 0.39361932$

$W_2 = 0.01995324$

The 2nd order result (with only 2 evaluation points) is accurate within 2 SF.

$$\int_{-\infty}^{\infty} e^{-x^2+x} dx = \int_{-\infty}^{\infty} e^{-x^2} e^{x} dx$$

For $N=2$: $$\int_{-\infty}^{\infty} e^{-x^2+x} dx \cong \frac{\sqrt{\pi}}{2} e^{\frac{1}{\sqrt{2}}} + \frac{\sqrt{\pi}}{2} e^{-\frac{1}{\sqrt{2}}} = \underline{2.234341}$$

For $N=5$: $$\int_{-\infty}^{\infty} e^{-x^2+x} dx \simeq W_0 e^{x_0} + W_1 e^{x_1} + W_1 e^{-x_1} + W_2 e^{x_2} + W_2 e^{-x_2} = \underline{2.275874}$$