Chapter 37 - Relativity

1. [37.51] The starships of the Solar Federation are marked with the symbol of the federation, a circle, hile starships of the Denebian Empire are marked with the empire's symbol, an ellipse whose major axis is 1.40 times longer than its minor axis. How fast, relative to an observer, does an empire ship have to travel for its marking to be confused with the marking of a federation ship?

   - Observed lengths contract, so the empire ship ought to move at speed v in the direction of its marking's major axis. Let the major and minor axes be a and b respectively, where $a = 1.40b$. We want:
   $$b = \frac{a}{\gamma} = a\sqrt{1 - \frac{v^2}{c^2}} \quad \Rightarrow \quad v = c\sqrt{1 - \frac{b^2}{a^2}} = 0.700c = 2.10 \times 10^8 \text{ m/s}$$

2. [37.59] In an experiment, two protons are shot directly toward each other, each moving at half the speed of light relative to the laboratory. (a) What speed does one proton measure for the other proton? (b) What would be the answer to part (a) if we used only non-relativistic Newtonian mechanics? (c) What is the kinetic energy of each proton as measured by (i) an observer at rest in the laboratory frame and (ii) an observer riding along with one of the protons? (d) What would be the ansers in part (c) if we used only non-relativistic Newtonian mechanics?

   - $v_1 = c/2$, $v_2 = -c/2$, $m_p = 1.67 \times 10^{-27}$ kg $= 938$ MeV
   - (a)
   $$v_{\text{rel}} = \frac{v_1 - v_2}{1 - v_1 v_2/c^2} = \frac{1}{1 + \frac{1}{4}} c = \frac{4}{5} c$$
   - (b)
   $$v_{\text{non-rel}} = v_1 - v_2 = c$$
   - (c)
   $$p = \gamma m v \quad , \quad \beta = \frac{v}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$
   $$\begin{aligned} T &= \sqrt{(pc)^2 + (mc^2)^2} - mc^2 = \sqrt{(\gamma m v c)^2 + (mc^2)^2} - mc^2 = mc^2\sqrt{1 + \gamma^2\beta^2} - mc^2 \\ &= mc^2\left[\sqrt{1 + \frac{\beta^2}{1 - \beta^2}} - 1\right] = mc^2\left[\sqrt{\frac{1}{1 - \beta^2}} - 1\right] = mc^2(\gamma - 1) \end{aligned}$$
   $$\text{(i)} \quad \beta = \frac{1}{2} \quad , \quad \gamma = \frac{2}{\sqrt{3}} \quad , \quad T_{(i)} = 0.15\, m_p c^2 = 2.31 \times 10^{-11} \text{ J} = 144 \text{ MeV}$$
   $$\text{(ii)} \quad \beta = \frac{4}{5} \quad , \quad \gamma = \frac{5}{3} \quad , \quad T_{(i)} = \frac{2}{3}\, m_p c^2 = 1.00 \times 10^{-10} \text{ J} = 625 \text{ MeV}$$
   - (d)
   $$\text{(i)} \quad T_{(i)} = \frac{1}{2} m_p (c/2)^2 = 1.88 \times 10^{-11} \text{ J} = 117 \text{ MeV}$$
   $$\text{(ii)} \quad T_{(i)} = \frac{1}{2} m_p c^2 = \times 7.51 \times 10^{-11} \text{ J} = 469 \text{ MeV}$$

3. [37.65] Two events observed in a frame of reference S have positions and times given by (x_1, t_1) and (x_2, t_2) respectively.

(a) Frame S' moves along the x-axis just fast enough that the two events occur at the same position in S'. Show that in S' the time interval $\Delta t'$ between the two events is given by:

$$\Delta t' = \sqrt{(\Delta t)^2 - \left(\frac{\Delta x}{c}\right)^2}$$

where $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$. Hence, show that if $\Delta x > c\Delta t$, there is no frame S' in which the two events occur at the same point. The interval $\Delta t'$ is sometimes called "the proper time interval" for the events. Is this term appropriate?

(b) Show that if $\Delta x > c\Delta t$, there is a different frame of reference $S'' \neq S'$ in which the two events occur simultaneously. Find the distance between the two events in S''; express your result in terms of $\Delta x, \Delta t, c$. This distance is sometimes called "proper length" Is this term appropriate?

(c) Two events are observed in a frame of reference S' to occur simultaneously at points separated by the distance of 2.50 m. In second frame S moving relative to S' along the line joining the two points in S', the two events appear to be separated by 5.00 m. What is the time interval between the events as measured in S?

- (a) Use Lorentz transformations:

$$x_1' = \gamma(x_1 + \beta c t_1) \quad , \quad c t_1' = \gamma(c t_1 + \beta x_1)$$
$$x_2' = \gamma(x_2 + \beta c t_2) \quad , \quad c t_2' = \gamma(c t_2 + \beta x_2)$$

The speed $v = \beta c$ of S' is such that $x_1' = x_2'$. Then, subtract the two vertical pairs of equations:

$$0 = \gamma(\Delta x + \beta c \Delta t) \quad , \quad c\Delta t' = \gamma(c\Delta t + \beta \Delta x)$$

Use the left equation to find β, and then substitute β (and $\gamma = 1/\sqrt{1-\beta^2}$) in the right equation:

$$\beta = -\frac{\Delta x}{c\Delta t} \quad \Rightarrow$$

$$\begin{aligned}\Delta t' &= \gamma\left(\Delta t + \frac{\beta \Delta x}{c}\right) = \frac{1}{\sqrt{1-(\frac{\Delta x}{c\Delta t})^2}}\left(\Delta t - \frac{\Delta x^2}{c^2 \Delta t}\right)\\ &= \frac{\Delta t}{\sqrt{1-(\frac{\Delta x}{c\Delta t})^2}}\left[1-\left(\frac{\Delta x}{c\Delta t}\right)^2\right] = \Delta t\sqrt{1-\left(\frac{\Delta x}{c\Delta t}\right)^2}\\ &= \sqrt{(\Delta t)^2 - \left(\frac{\Delta x}{c}\right)^2}\end{aligned}$$

If $\Delta x > c\Delta t$, then one would obtain an imaginary $\Delta t'$, which is unphysical.

- (b) Now, $t_1' = t_2'$. We use the same approach:

$$\Delta x' = \gamma(\Delta x + \beta c \Delta t) \quad , \quad 0 = \gamma(c\Delta t + \beta \Delta x)$$

Eliminate β from the right equation...

$$\beta = -\frac{c\Delta t}{\Delta x} \quad \Rightarrow$$

$$\Delta x' = \gamma(\Delta x + \beta c\Delta t) = \frac{1}{\sqrt{1-(\frac{c\Delta t}{\Delta x})^2}}(\Delta x - \frac{c^2\Delta t^2}{\Delta x}) = \sqrt{(\Delta x)^2 - (c\Delta t)^2}$$

- (c) Use part (b): $\Delta x' = 2.5$ m (where the events are simultaneous), $\Delta x = 5.0$ m

$$\Delta x' = \sqrt{(\Delta x)^2 - (c\Delta t)^2} \quad \Rightarrow \quad \Delta t = \frac{(\Delta x)^2 - (\Delta x')^2}{c} = 1.4 \times 10^{-8}\ \text{s}$$