Chapter 36 - Diffraction

Fraunhofer difraction pattern

Monochromatic light of wavelength λ is normally incident on the narrow slit of witdh a. Let the coordinate system origin O be at the slit's center. The screen is placed a horisontal distance x away from the slit, and the brightest diffraction spot will be at the point S with coordinates $(x, 0)$. Consider a point P on the screen whose coordinates are (x, y). We will use the ray that originates from the slit's center O as a reference. Any other ray that originates from the place $(0, y')$ in the slit is phase-shifted from the reference ray by $\phi \approx ky' \sin\theta$, where $\theta = \measuredangle\text{POS}$ $(\tan\theta = y/x)$ and $k = 2\pi/\lambda$ is the light's wavenumber. The total electric field at P is the sum of oscillating electric fields (of the same unknown amplitude) that originate from all points $(0, y')$ within the slit:

$$E(y) \propto \int\limits_{-a/2}^{a/2} dy' \, \cos(ky' \sin\theta) = \frac{\sin(ky' \sin\theta)}{k \sin\theta}\Big|_{-a/2}^{a/2} = \frac{2\sin\left(\frac{a}{2} k \sin\theta\right)}{k \sin\theta}$$

More precisely, this is the amplitude of the total electric field at P. The actual electric field is oscillatory, but we could use the trigonometric formula:

$$\cos(ky' \sin\theta - \omega t) = \cos(ky' \sin\theta)\cos(\omega t) + \sin(ky' \sin\theta)\sin(\omega t) \xrightarrow{\theta \ll 1} \cos(ky' \sin\theta)\cos(\omega t)$$

inside the integral and obtain the above result multiplied by the oscillatory part $\cos(\omega t)$ after integrating y'. The integral itself was solved by:

$$\int d\xi \, \cos(q\xi) = \frac{1}{q} \int d(q\xi) \, \cos(q\xi) = \frac{1}{q} \sin(q\xi) + \text{const}$$

The light intensity is $I \propto E^2$, so that:

$$I(y) = I_0 \left[\frac{\sin\left(\frac{\pi a}{\lambda} \sin\theta\right)}{\frac{\pi a}{\lambda} \sin\theta} \right]^2$$

Note that the above result for $E(y)$ has an unknown amplitude, but does predict a θ-dependence of the kind $\sin(A \sin\theta)/B \sin\theta$ where A, B are some constants. The square of this must also appear in $I(y)$, but here we can introduce a "known" light intensity I_0 at the center $(x, 0)$ of the brightest diffraction peak because the functioin $\sin(\alpha)/\alpha$ has the well-known property:

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$$

We, hence, have the brightest spot at $\theta = 0$. Other difraction maximums are at non-zero angles θ such that:

$$\sin\left(\frac{\pi a}{\lambda} \sin\theta\right) = \pm 1 \quad \Rightarrow \quad \frac{\pi a}{\lambda} \sin\theta = \left(m + \frac{1}{2}\right)\pi \quad , \quad m = 0, \pm 1, \pm 2, \dots$$

Darkest spots are at non-zero angles θ such that:

$$\sin\left(\frac{\pi a}{\lambda} \sin\theta\right) = 0 \quad \Rightarrow \quad \frac{\pi a}{\lambda} \sin\theta = m\pi \quad , \quad m = \pm 1, \pm 2, \dots$$

1. [36.57] A slit 0.360 mm wide is illuminated by parallel rays of light that wave a wavelength of 540 nm. The diffraction pattern is observed on the screen that is 1.20 m away from the slit. The intensity at the center of the central maximum ($\theta = 0$) is I_0. (a) What is the distance on the screen from the center of the central maximum to the first minimum? (b) What is the distance on the screen from the center of the central maximum to the point where the intensity has fallen to $I_0/2$?

- $a = 0.360$ mm, $\lambda = 540$ nm, $x = 1.20$ m

$$I(\theta) = I_0 \left[\frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a}{\lambda}\sin\theta} \right]^2 \to 0 \quad \Rightarrow \quad \frac{\pi a}{\lambda}\sin\theta = m\pi \quad , \quad m = \pm 1, \pm 2, \ldots$$

- (a) Approximatelly, $y/x = \tan\theta \approx \sin\theta$:

$$m = 1 \quad \Rightarrow \quad \sin\theta = \frac{\lambda}{a} \quad \Rightarrow \quad \tan\theta \approx \sin\theta = \frac{\lambda}{a} = \frac{y}{x} \quad \Rightarrow \quad y = \frac{\lambda x}{a} = 1.8 \text{ mm}$$

- (b) Solve numerically $\sin(\xi)/\xi = 1/\sqrt{2}$:

$$\frac{I(\theta)}{I_0} = \frac{1}{2} = \left[\frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a}{\lambda}\sin\theta} \right]^2 \approx \left[\frac{\sin\left(\frac{\pi a}{\lambda}\frac{y}{x}\right)}{\frac{\pi a}{\lambda}\frac{y}{x}} \right]^2$$

$$\frac{\sin\left(\frac{\pi a}{\lambda}\frac{y}{x}\right)}{\frac{\pi a}{\lambda}\frac{y}{x}} \approx \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \frac{\pi a}{\lambda}\frac{y}{x} = 1.39 \quad \Rightarrow \quad y = \frac{1.39\lambda x}{\pi a} = 7.96 \times 10^{-4} \text{ m}$$

2. [36.59] Consider N evenly spaced narrow slits. Use the small-angle approximation $\sin\theta \approx \theta$ (for θ in radians) to prove the following: For an intensity maximum that occurs at an an angle θ, the intensisy minima immediatelly adjacent to it are at angles $\theta \pm \lambda/Nd$, so that the angular width of the principal maximum is $2\lambda/Nd$. This is proportional to $1/N$ on the basis of energy conservation.

   - Narrow slits means $a < \lambda$ where a is the slit width: diffraction is not seen since the brightest peak spreads out to an angle "greater" than 180^o. We are dealing with an interference of N equally-spaced point-like sources (in the limit $a \ll \lambda$). Let d be the separation between two adjacent sources. The phase difference of light rays from two adjacent sources observed at an angle θ on the screen is $\phi = 2\pi(d/\lambda)\sin\theta$. We must add oscillatory electric fields from all slits to obtain the total field at the observation point (angle θ):

$$E(\theta) = E_0 \sum_{n=0}^{N-1} \cos(n\phi - \omega t) = E_0 \sum_{n=0}^{N-1} \cos\left(2\pi n \frac{d\sin\theta}{\lambda} - \omega t\right)$$

These sums are most easily solved using "phasors", i.e. using the Euler formula $e^{ix} = \cos x + i \sin x$ for complex numbers:

$$\begin{aligned} \sum_{n=0}^{N-1} \cos(n\phi - \omega t) &= \text{Re}\left\{ \sum_{n=0}^{N-1} e^{i(n\phi - \omega t)} \right\} = \text{Re}\left\{ e^{-i\omega t}\left[1 + e^{i\phi} + e^{2i\phi} + \cdots + e^{(N-1)i\phi}\right]\right\} \\ &= \text{Re}\left\{ e^{-i\omega t}\left[1 + z + z^2 + \cdots + z^{N-1}\right]\right\} \end{aligned}$$

We have labeled $z = e^{i\phi}$ and Re stands for "real part". Then, define:

$$S_N = 1 + z + z^2 + \cdots + z^{N-1}$$

and note:

$$zS_N = z + z^2 + \cdots + z^{N-1} + z^N$$

$$S_N - zS_N = (1-z)S_N = 1 - z^N \quad \Rightarrow \quad S_N = \frac{1 - z^N}{1 - z}$$

Therefore:

$$\sum_{n=0}^{N-1} \cos(n\phi - \omega t) = \text{Re}\left\{e^{-i\omega t} S_N\right\} = \text{Re}\left\{ e^{-i\omega t}\frac{1 - z^N}{1 - z} \right\} = \text{Re}\left\{ e^{-i\omega t}\frac{1 - e^{iN\phi}}{1 - e^{i\phi}} \right\}$$

Using again Euler's formula, we find:

$$e^{ix} = \cos x + i \sin x \quad , \quad e^{-ix} = \cos x - i \sin x \quad \Rightarrow \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$1 - e^{i\phi} = e^{i\phi/2}(e^{-i\phi/2} - e^{i\phi/2}) = -2ie^{i\phi/2} \sin(\phi/2)$$
$$1 - e^{Ni\phi} = e^{iN\phi/2}(e^{-iN\phi/2} - e^{iN\phi/2}) = -2ie^{iN\phi/2} \sin(N\phi/2)$$

and hence:

$$\sum_{n=0}^{N-1} \cos(n\phi - \omega t) = \text{Re}\left\{e^{-i\omega t} \frac{e^{iN\phi/2}}{e^{i\phi/2}} \frac{\sin(N\phi/2)}{\sin(\phi/2)}\right\} = \frac{\sin(N\phi/2)}{\sin(\phi/2)} \times \text{Re}\left\{e^{i(N-1)\phi/2} e^{-i\omega t}\right\}$$
$$= \frac{\sin(N\phi/2)}{\sin(\phi/2)} \times \cos\Big((N-1)\phi/2 - \omega t\Big)$$

The intensity of light is proportional to the electric field *amplitude* squared:

$$I(\theta) \propto |E(\theta)|^2 = \frac{\sin(N\phi/2)}{\sin(\phi/2)}$$

We re not interested in the oscillatory part that has ωt...

- Now recall $\phi = 2\pi(d/\lambda)\sin\theta \approx 2\pi(d/\lambda)\theta$. Dark fringes are obtained at places where $I \to 0$, i.e:

$$\sin(N\phi/2) = 0 \quad \Rightarrow \quad N\phi/2 = m\pi \quad \Rightarrow \quad \theta \approx \frac{\lambda}{2\pi d}\phi = \frac{\lambda}{2\pi d}\frac{2\pi m}{N} = \frac{\lambda}{Nd} m \quad , \quad m \in \mathbb{Z}$$

The exception are the places where we would find:

$$I(\theta) \propto \frac{\sin(N\phi/2)}{\sin(\phi/2)} \to \frac{0}{0} \quad \Rightarrow \quad \phi/2 = m\pi \quad \Rightarrow \quad \theta \approx \frac{\lambda}{2\pi d}\phi = \frac{\lambda}{d} m \quad , \quad m \in \mathbb{Z}$$

These are bright fringes. Clearly, the angular separation between two adjacent dark fringes is $2 \times \lambda/Nd$.

3. [36.63] What is the longest wavelength that can be observed in the third order for a transmission grating having 9200 slits/cm? Assume normal incidence.

- The distance between adjacent slits is $d = 1/9200$ cm, i.e. $d = 1.087$ μm. The light of wavelength λ has bright fringes at angles θ such that $d \sin\theta = m\lambda$ for integer m. This integer defines the "order" of the bright difraction maximum. Since we are told to fix $m = 3$, increasing λ must correspond to increasing $\sin\theta$. However, the latter is bounded by 1, so the longest wavelength is:

$$\lambda_{\text{max}} = \frac{d \times 1}{m} = \frac{d}{3} = 362 \text{ nm}$$

4. [36.69] An astronaut in the space shuttle can just resolve two point sources on earth that are 65.0 m apart. Assume that the resolution is diffraction-limited and use Rayleigh's criterion. What is the astronaut's altitude above the earth. Treat his eye (pupil) as a circular aperture with diameter of 4.00 mm, and take the wavelength of light to be 550 nm. Ignore the effect of fluid in the eye.

- Rayleigh's criterion is that one can resolve two point-like sources of light only up the point when the central diffraction peak of one coincides with the first diffraction minimum of the other source. For a circular aperture of diameter D, the first diffraction minimum is found at the angle θ_1 given by:

$$\sin\theta_1 = 1.22\frac{\lambda}{D}$$

So, the two point-like sources on earth must have angular separation of at least θ_1 to be resolved. If the astronaut's altitude is h is much larger than the distance $l = 65.0$ m between the sources, we may approximate $l/h = \tan\theta \approx \sin\theta$ and find:

$$\sin\theta_1 = 1.22\frac{\lambda}{D} \approx \frac{l}{h} \quad \Rightarrow \quad h \approx \frac{lD}{1.22\lambda} = 387 \text{ km}$$