Chapter 34 - Geometric Optics

1. [34.89] A glass rod with a refractive index of 1.55 is ground and polished at both ends to hemispherical surfaces with radii 6 cm. When an object is placed on an axis of the rod 25.0 cm to the left of the left-hand end, the final image is formed 65.0 cm to the right of the right-hand end. What is the length of the rod measured between the vertices of the two hemispherical surfaces?

   - $n = 1.55$, $|R| = 6$ cm, $s_1 = 25.0$ cm, $s_2' = 65.0$ cm

   $$\frac{1}{s_1} + \frac{n}{s_1'} = \frac{n-1}{R_l} \qquad \text{(left hemisphere)}$$

   $$\frac{n}{s_2} + \frac{1}{s_2'} = \frac{1-n}{R_r} \qquad \text{(right hemisphere)}$$

   $$s_2 = l - s_1'$$

   The last equation says that the left hemisphere's image is the right hemisphere's object. From the first equation we find:

   $$s_1' = \frac{n}{\frac{n-1}{R_l} - \frac{1}{s_1}}$$

   Substituting in the other two equatiions we get:

   $$\frac{n}{l - s_1'} + \frac{1}{s_2'} = \frac{1-n}{R_r}$$

   $$l = s_1' + \frac{n}{\frac{1-n}{R_r} - \frac{1}{s_2'}} = \frac{n}{\frac{n-1}{R_l} - \frac{1}{s_1}} + \frac{n}{\frac{1-n}{R_r} - \frac{1}{s_2'}}$$

   The text of the problem is not clear about how the hemispheres are made: convex or concave on either end. So, let's compute all possibilities:

   $$\frac{n}{\frac{n-1}{R_l} - \frac{1}{s_1}} = \left\{ \begin{array}{ccc} 30.0 \text{ cm} & , & R_l = +R \\ -11.8 \text{ cm} & , & R_l = -R \end{array} \right\} \quad , \quad \frac{n}{\frac{1-n}{R_r} - \frac{1}{s_2'}} = \left\{ \begin{array}{ccc} -14.5 \text{ cm} & , & R_r = +R \\ 20.3 \text{ cm} & , & R_r = -R \end{array} \right\}$$

   $$l = \left\{ \begin{array}{ccc} 15.5 \text{ cm} & , & (R_l, R_r) = (+,+)R \\ 50.3 \text{ cm} & , & (R_l, R_r) = (+,-)R \\ -26.3 \text{ cm} & , & (R_l, R_r) = (-,+)R \\ 8.5 \text{ cm} & , & (R_l, R_r) = (-,-)R \end{array} \right\}$$

   The negative rod length l makes no sense, but the other results could work. $l = 50.3$ cm corresponds to, perhaps, the most natural assumption that both hemispheres are bulging out of the rod - and it is the official solution from the textbook.

2. [34.93] A convex mirror and a concave mirror are placed on the same optical axis, separated by the distance $L = 0.600$ m. The radius of curvature of each mirror has the magnitude of 0.360 m. A light source is located between the mirrors a distance x from the concave mirror. (a) What distance x will result in the rays from the source returned to the source after reflecting first from the convex mirror and then from the concave mirror? (b) Repeat the part (a), but now let the rays reflect first from the concave mirror and then from the convex one.

   - $L = 0.600$ m, $R = 0.360$ m, $R_l = -R$ (convex to the left of the light source), $R_r = +R$ (concave to the right of the light source); the choice to put the convex mirror on the left is arbitrary, but must be applied consistently...

- (a) Let s be the distance of the image from the left (convex, $s < 0$) mirror's vertex after the first reflection from the left mirror. Then:

$$\frac{1}{L-x} + \frac{1}{s} = \frac{2}{R_l} = -\frac{2}{R} \quad \Rightarrow \quad s = \frac{1}{-\frac{2}{R} - \frac{1}{L-x}}$$

$$\frac{1}{L-s} + \frac{1}{x} = \frac{2}{R_r} = \frac{2}{R} \quad \Rightarrow \quad L - s = \frac{1}{\frac{2}{R} - \frac{1}{x}}$$

Add the equations on the right and solve for x:

$$L = \frac{1}{\frac{2}{R} - \frac{1}{x}} - \frac{1}{\frac{2}{R} + \frac{1}{L-x}}$$

$$\frac{1}{L} = \frac{\left(\frac{2}{R} + \frac{1}{L-x}\right)\left(\frac{2}{R} - \frac{1}{x}\right)}{\frac{1}{x} + \frac{1}{L-x}} = \frac{\left(\frac{2}{R}\right)^2 - \frac{2}{R}\left(\frac{1}{x} - \frac{1}{L-x}\right) - \frac{1}{x(L-x)}}{\frac{1}{x} + \frac{1}{L-x}} = \frac{\left(\frac{2}{R}\right)^2 - \frac{2}{R}\frac{L-2x}{x(L-x)} - \frac{1}{x(L-x)}}{\frac{L}{x(L-x)}}$$

$$1 = \left(\frac{2}{R}\right)^2 x(L-x) - \frac{2}{R}(L-2x) - 1$$

$$\cdots$$

$$x^2 - (R+L)x + \frac{R(R+L)}{2} = 0$$

$$x = \frac{R + L \pm \sqrt{L^2 - R^2}}{2} = \frac{\sqrt{L+R}}{2}\left(\sqrt{L+R} \pm \sqrt{L-R}\right)$$

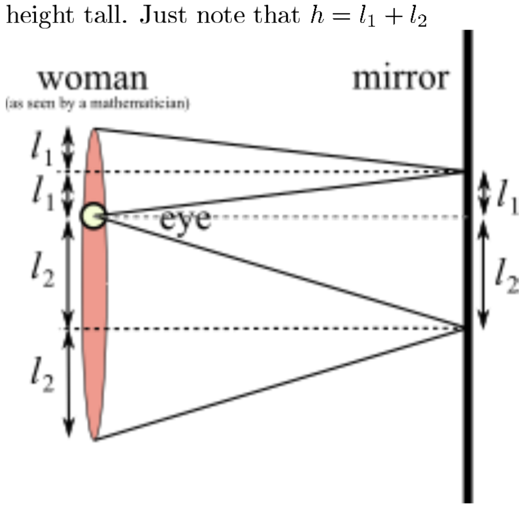
$$x = 0.72 \text{ m} \quad \text{or} \quad x = 0.24 \text{ m}$$

The result $x = 0.72$ m has to be discarded because it is larger than the distance $L = 0.6$ m between the mirrors. Therefore, the final answer is $x = 0.24$ m.

- (b) No need for a new calculatiion. The source and the image are reversed with respect to the part (a). Since the image and source are required to coincide, we again find $x = 0.24$ m.

3. [34.71] What is the size of the smallest vertical plane mirror in which a woman of height h can see her full-length image?

- It's very easy to see from a picture that the mirror needs to be at least a half of the woman's height tall. Just note that $h = l_1 + l_2$



4. [34.73] A concave mirror is to form an image of the filament of a headlight lamp on a screen 8.00 m from the mirror. The filament is 6.00 mm tall, and the image is to be 24.0 cm tall. (a) How far in front of the vertex of the mirror should the filament be placed? (b) What should be the radius of curvature of the mirror?

   - $s' = 8.00$ m, $y = 0.00600$ m, $y' = -0.24$ m, $s = ?$, $R > 0$

$$\frac{y'}{y} = -\frac{s'}{s} \quad \Rightarrow \quad s = -\frac{s'y}{y'} = 0.2 \text{ m}$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} \quad \Rightarrow \quad R = \frac{2}{\frac{1}{s} + \frac{1}{s'}} = 0.390 \text{ m}$$

5. [34.81] What should be the index of refraction of a transparent sphere in order for paraxial rays from an infinitely distant object to be brought to a focus at the vertex of the surface opposite the point of incidence?

   - ...

$$\frac{1}{\infty} + \frac{n}{2R} = \frac{n-1}{R} \quad \Rightarrow \quad n = 2n - 2 \quad \Rightarrow \quad n = 2$$

6. [34.96] Prove that when to thin lenses with focal lengths f_1 and f_2 are placed in contact, the focal length f of the combination is given by the relationship:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

- Let an object be a distnce s away from the lenses. It forms an image t on the lens with focal length f_1. This becomes the object for the second lens, but nominally a distance $-t$ away. Then:

$$\frac{1}{s} + \frac{1}{t} = \frac{1}{f_1} \quad , \quad \frac{1}{-t} + \frac{1}{s'} = \frac{1}{f_2} \quad \Rightarrow \quad \frac{1}{s} + \frac{1}{s'} = \frac{1}{f_1} + \frac{1}{f_2} \equiv \frac{1}{f}$$

1. [34.99] When an object is placed at the proper distance to the left of a converging lens, the image is focused on a screen 30.0 cm to the right of the lens. A diverging lens is now placed 15.0 cm to the right of the converging lens, and it is found that the screen must be moved 19.2 cm farther to the right to obtain a sharp image. What is the focal length of the diverging lens?

   - The image is focused if it appears in the focal plane. So, $f = 30.0$ cm is the focal length of the lens ($f > 0$ because the lens is converging). Let the added diverging lens, places a distance $l = 15.0$ cm from the converging lens, have focal length $f' < 0$. The image of an infinitely distant object on the converging lens is at its focal point, now $s = l - f = -15.0$ cm away from the diverging lens. Hence, the final image is the distance s' from the diverging lens according to:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f'}$$

   This is where the sharp image now appears:

$$s' = f - l + 19.2 \text{ cm} = 34.2 \text{ cm}$$

   Therefore:

$$f' = \frac{1}{\frac{1}{s} + \frac{1}{s'}} = -26.7 \text{ cm}$$