Chapter 33 - The Nature and Propagation of Light

1. [33.39] An inside corner of a cube is lined with mirrors to make a corner reflector. A ray of light is reflected successively from each of the three mutually perpendicular mirrors. Show that the ray's final direction is always exactly opposite to its initial direction.

   - Align the cube edges with x, y, z axes of the Cartesian coordinate system. Specify the incident direction of the ray by a unit vector $\hat{\mathbf{n}}_0 = (n_x, n_y, n_z)$. When the ray reflects from the cube's face that lies in the xy-plane, the perpendicular n_z component of its direction changes sign, yielding $\hat{\mathbf{n}}_1 = (n_x, n_y, -n_z)$. When the ray reflects next from the cube's face that lies in the yz-plane, the perpendicular n_x component of its direction changes sign, yielding $\hat{\mathbf{n}}_2 = (-n_x, n_y, -n_z)$. Lastly, after the reflection from the zx-plane we get the final ray's direction $\hat{\mathbf{n}}_3 = (-n_x, -n_y, -n_z) = -\hat{\mathbf{n}}_0$. It doesn't matter in which order the ray hits the three orthogonal faces of the cube, and what exactly the original direction $\hat{\mathbf{n}}$ is.

2. [33.43] A ray of light is incident in air on the horizontal top surface of a block (rectangular prism) made from a transparent solid whose index of refraction is $n = 1.38$. What is the largest angle of incidence θ_a for which total internal reflection will occur at the vertical face?

   - The minimum incidence angle θ_t for total reflection is:

   $$n \sin\theta_t = 1 \times \sin 90^o = 1 \qquad \Rightarrow \qquad \theta_t = \arcsin\left(\frac{1}{n}\right) = 46.5^o$$

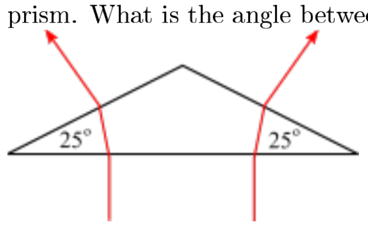
   A ray incident on the vertical wall at this angle is incident on the horizontal wall at $\theta_i = 90^o - \theta_t = 43.6^o$ inside the prism. This refracted ray comes enters the prism from the outside at the angle θ_a:

   $$1 \times \sin\theta_a = n \sin\theta_i = n\cos\theta_t \qquad \Rightarrow \qquad \theta_a = \arcsin(n\cos\theta_t) = \arcsin\left(n\sqrt{1 - \frac{1}{n^2}}\right) = \arcsin\left(\sqrt{n^2 - 1}\right) = 72.1^o$$

   This is the largest angle that allows total reflection (draw a picture to convince yourself).

3. [33.45] A ray of light travelling in a block of glass ($n = 1.52$) is incident on the top surface at an angle of 57.2^o with respect to the normal in the glass. If a layer of oil is placed on the top surface of the glass, the ray is totally reflected. What is the maximum possible index of refraction of the oil?

4. [33.53] A triangular prism has a refractive index of 1.66. Two light rays are parallel as they enter the prism. What is the angle between them as they emerge?



   - If one beam deflects by angle δ, then the two beams diverge by 2δ due to symmetry. The beams enter the prism perpendicular to the first prism-air interface, so they don't change direction there. The incidence angle on the second interface (a leg of the isosceles triangle) is $\theta_i = 25^o$, equal to the base angle of the isosceles triangle. The refraction angle θ_o is:

   $$n \sin\theta_i = 1 \times \sin\theta_o \qquad \Rightarrow \qquad \theta_o = \arcsin(n \sin\theta_i) = 44.6^o$$

This angle is measured with respect to the perpendicular direction on the triangle's leg. We need to rotate it by 25^o in order to find the deflection angle from the original beams' vertical direction:

$$\delta = \theta_o - 25^o = 19.6^o$$

Hence, the beams diverge by:

$$2\delta \approx 39.1^o$$

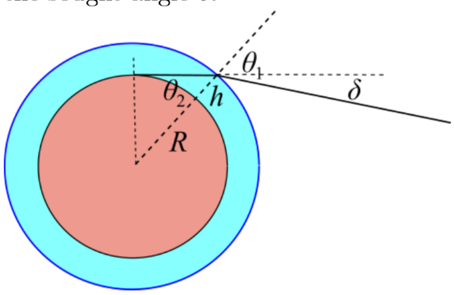
in the opposite directions than the picture suggests...

5. [33.55] When the sun is either raising or setting and appears to be just on the horizon, it is in fact below the horizon. The explanation for this seeming paradox is that light from the sun bends slightly when entering earth atmosphere. Since our perception is based on the idea that light propagates in straight lines, we perceive the light to be coming from an apparent position that is an angle δ above the sun's true position. (a) Make the simplifying assumptions that the atmosphere has uniform density and hence a uniform index of refraction n, and extends to a height h above the earth's surface at which point it abruptly stops. Show that the angle δ is given by:

$$\delta = \arcsin\left(\frac{nR}{R+h}\right) - \arcsin\left(\frac{R}{R+h}\right)$$

where R is the radius of the earth. (b) Calculate δ using $n = 1.0003$, $h = 20$ km, $R = 6378$ km. How does this compare to the angular radius of the sun, which is about one quarter of a degree?

- See the figure for the relationship between the sunlight incident angle θ_1, refraction angle θ_2 and the sought angle δ:



$$\theta_1 = \theta_2 + \delta \quad , \quad \sin\theta_2 = \frac{R}{R+h}$$

Snell's law completes the system of equations:

$$1 \times \sin\theta_1 = n \sin\theta_2$$

We find:

$$\sin(\theta_2 + \delta) = \frac{nR}{R+h}$$

$$\delta = \arcsin\left(\frac{nR}{R+h}\right) - \theta_2 = \arcsin\left(\frac{nR}{R+h}\right) - \arcsin(\sin\theta_2) = \arcsin\left(\frac{nR}{R+h}\right) - \arcsin\left(\frac{R}{R+h}\right)$$

6. [33.63] A beam of unpolarized sunlight strikes the vertical plastic wall of a water tank at an unknown angle. Some of the light reflects from the wall and enters the water. The refractive index of the plastic wall is $n = 1.61$. If the reflected light is observed to be completely polarized, what angle does this beam make with the normal inside the water?

- The refraction index of water is $n_w = 1.33$. The reflected and refracted beams on the wall are perpendicular, so the angle θ_{air} of incidence on the wall (with respect to the parpendicular direction on the wall) is:

$$1 \times \sin\theta_{\text{air}} = n \sin\theta_{\text{wall}} = n\cos\theta_{\text{air}} \quad \Rightarrow \quad \theta_{\text{air}} = \arctan(n) = 58.2^o$$

The incidence angle of the reflected ray on the water surface is then:

$$\theta'_{\text{air}} = 90^o - \theta_{\text{air}} = 31.8^o$$

This beam gets refracted in the water:

$$1 \times \sin\theta'_{\text{air}} = 1 \times \cos\theta_{\text{air}} = n_w \sin\theta_w \quad \Rightarrow \quad \theta_w = \arcsin\left(\frac{1}{n_w}\cos\theta_{\text{air}}\right) = 23.3^o$$

7. [33.57] A ray of light goes from point A in which the speed of light is v_1 to point B in a medium in which the speed of light is v_2. The ray strikes the interface a horizontal distance x to the right of the point A. (a) Show that the time required for the light to go from A to B is:

$$t = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (l-x)^2}}{v_2}$$

where h_1 and h_2 are vertical distances of A and B from the interface respectively, and l is the horizontal distance between A and B. (b) Take the derivative of t with respect to x. Set the derivative to zero to show that t reaches its minimum value when $n_1 \sin\theta_1 = n_2 \sin\theta_2$. This is Snell's law and corresponds to the actual path taken by light. This is an example of Fermat's principle of least time: light takes the path between two points A and B that minimizes its travel time.

- (a) is obvious when you draw a picture: $\sqrt{h_1^2 + x^2}$ is the distance the ray travels from A to the interface, with speed v_1, etc...
- (b) Note that:

$$\sin\theta_1 = \frac{x}{\sqrt{h_1^2 + x^2}} \quad , \quad \sin\theta_2 = \frac{l-x}{\sqrt{h_2^2 + (l-x)^2}}$$

Then:

$$\begin{aligned} \frac{dt}{dx} &= \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{l-x}{v_2\sqrt{h_2^2 + (l-x)^2}} \to 0 \\ \frac{\sin\theta_1}{v_1} &= \frac{\sin\theta_2}{v_2} \\ \frac{c}{v_1}\sin\theta_1 &= \frac{c}{v_2}\sin\theta_2 \\ n_1\sin\theta_1 &= n_2\sin\theta_2 \end{aligned}$$