PHYS 262

George Mason University

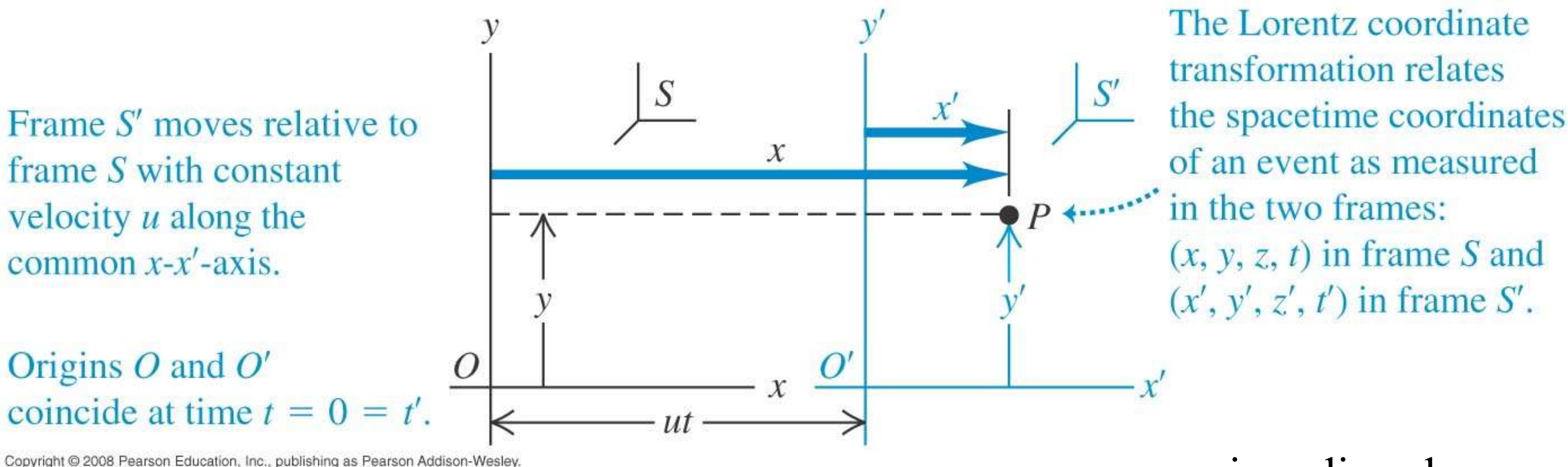
Prof. Paul So

Chapter 37: Relativity

- Events and Inertial Reference Frames
- Principles of Einstein's Special Relativity
- Relativity of Simultaneity, Time Intervals, Length
- Lorentz Transformation
- Relativistic Momentum & Energy
- Relativistic Doppler Shift

Lorentz Coordinate Transformation

Transforming the space-time coordinates from S to S' correctly so that physical laws satisfying SR are invariant.



u in x-dir only.

$$(x, y, z, t) \quad \longleftrightarrow \quad (x', y', z', t')$$

Lorentz Transformation

$$S \to S' \begin{cases} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{u}{c^2}x\right) \end{cases}$$

$$S' \to S \begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \frac{u}{c^2}x'\right) \end{cases}$$

Lorentz Velocity Transform

From the principle of relativity, there should be no physical distinction for the two inertial observers in relative motion.

So the Lorentz Velocity Transform equation and its inverse transform should have the same form but with u ←→ $-u$ for the inverse transform of $\boldsymbol{v}$ in term of $\boldsymbol{v}$'.

$$\begin{cases} v_x = \dfrac{v'_x + u}{1 + \dfrac{u}{c^2} v'_x} \\ v_y = \dfrac{v'_y}{\gamma\left(1 + \dfrac{u}{c^2} v'_x\right)} \end{cases} \qquad \begin{cases} v'_x = \dfrac{v_x - u}{1 - \dfrac{u}{c^2} v_x} \\ v'_y = \dfrac{v_y}{\gamma\left(1 - \dfrac{u}{c^2} v_x\right)} \end{cases}$$

Relativistic Momentum & Energy

As we have seen, time intervals, length intervals, and velocity change according to the Lorentz Transformation depending on the observer's frame of reference.

$$(x,t) \quad \leftrightarrow \quad (x',t') \quad \text{Lorentz Transformation}$$

Other *dynamical quantities* (such as momentum, energy, etc.) must also be appropriately expressed so that the laws of physics satisfy the following conditions:

- Satisfy the two postulates of Special Relativity:
  - Laws of physics (e.g., conservation of momentum, conservation of energy, Newton's laws) apply equally to all inertial observers.
  - Speed of light in vacuum same for all inertial observers
- The modified relativistic dynamical quantities should reduce to the classical ones for $u \ll c$.

Relativistic Momentum & Energy

Relativistic Momentum:

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

Momentum of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in the lab frame (*S*-frame).

Relativistic Energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Total energy of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in lab frame (*S*-frame).

Total Relativistic Energy

Note:

$$\vec{\mathbf{P}} = \gamma m \vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}} \qquad E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

→The mass "*m*" which we have been using is a *constant* in our analysis. It is called the **rest mass** ("proper" mass) and is the mass of an object measured by an observer *stationary* with the object.

→The quantity $m_{\text{rel}} = \gamma m$ is called the "relativistic mass" and is *not* a constant for a moving object and is measured by an observer *not* at rest with the object.

Total Relativistic Energy

For a particle at rest in a reference frame, it will have $v = 0\ m/s$ and $\gamma = 1$

$$E = \gamma mc^2 = mc^2$$

So, there is a *residual* Total Relativistic Energy $E = mc^2 \neq 0$ even for a particle *at rest*. The quantity mc^2 is called the **Rest Energy**.

Rest Energy (mc^2) →

- Independent of velocity
- Proportional to the mass of the particle
- *Mass is a form of energy*

Total Relativistic Energy

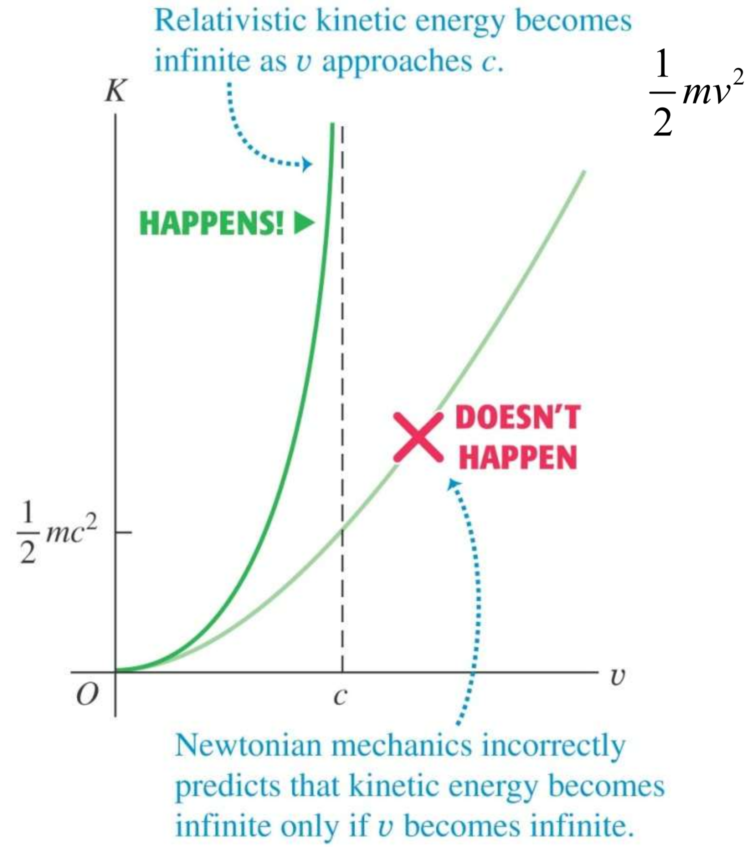
For a particle in motion, we can definite the relativistic Kinetic Energy as the difference between the **Total Relativistic Energy** and the **Rest Energy,**

$$KE = E - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

total energy (E)

rest energy (mc^2)

Note for $v \rightarrow c$, $KE \rightarrow \infty$



Relativistic KE → Classical KE

Slow moving particle regime $v << c$, laws of physics should be unchanged !

Using binomial theorem, $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1+\left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right)+O\left(\left(\frac{v^2}{c^2}\right)^2\right) \cong 1+\frac{1}{2}\frac{v^2}{c^2}$

Substituting this into the equation for Relativistic KE,

$$KE = \left(1-\frac{v^2}{c^2}\right)^{-1/2} mc^2 - mc^2 = \left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right] mc^2$$

$$\cong \left[1+\frac{1}{2}\frac{v^2}{c^2}-1\right] mc^2$$

$$= \frac{1}{2}mv^2$$

This is the classical result for $v << c$.

Total Relativistic Energy

Since $E = \gamma mc^2$ is the **total** relativistic energy of the system,

→ ***E* is *conserved* in *all* processes !**

- It combines with the two classical independent conservation laws:
  - conservation of energy
  - conservation of mass
- The statement on the Conservation of Total Relativistic Energy is more *general*

Conservation Laws

The following set of equations form the generalized conservation laws in Special Relativity.

$$\begin{cases} E_i = E_f & \text{Conservation of Relativisity Energy} \\ \vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f & \text{Conservation of Relativistiy Momentum} \end{cases}$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

AND, these conservation laws apply to *all* processes equally in *all* inertial reference frames !

New Energy Units

Electron Volt (eV):

The energy equals to moving one positive test charge *e*+ (1 Coulomb) across an electric potential of 1 volt.

$$1eV = (1.6022\times10^{-19}C)(1V) = 1.6022\times10^{-19}J$$

Example: Rest Mass Energy of an electron $m_e = 9.109\times10^{-31}kg$

$$E_0 = m_e c^2 = 9.109\times10^{-31}kg\left(2.997\times10^{8}m/s\right)^2$$

$$= 8.18171\times10^{-14}J\left(\frac{1eV}{1.6022\times10^{-19}J}\right) = 5.11\times10^{5}eV = 0.511MeV$$

$m_e = 0.511MeV/c^2$ ← (mass of *e* in units of *eV* and *c*)

Energy-Momentum Relation

$$E = \gamma mc^2$$

$$E^2 = \gamma^2 m^2 c^4$$

$$p = \gamma mv$$

$$c^2 p^2 = \gamma^2 m^2 v^2 c^2$$

$\vdots$

Subtract and Simiplify

$$E^2 - c^2 p^2 = m^2 c^4$$

Energy-Momentum Relation

$$E^2 - c^2 p^2 = m^2 c^4$$

This means that the combination $E^2 - c^2 p^2$ is independent of motion and is an *invariant* quantity,

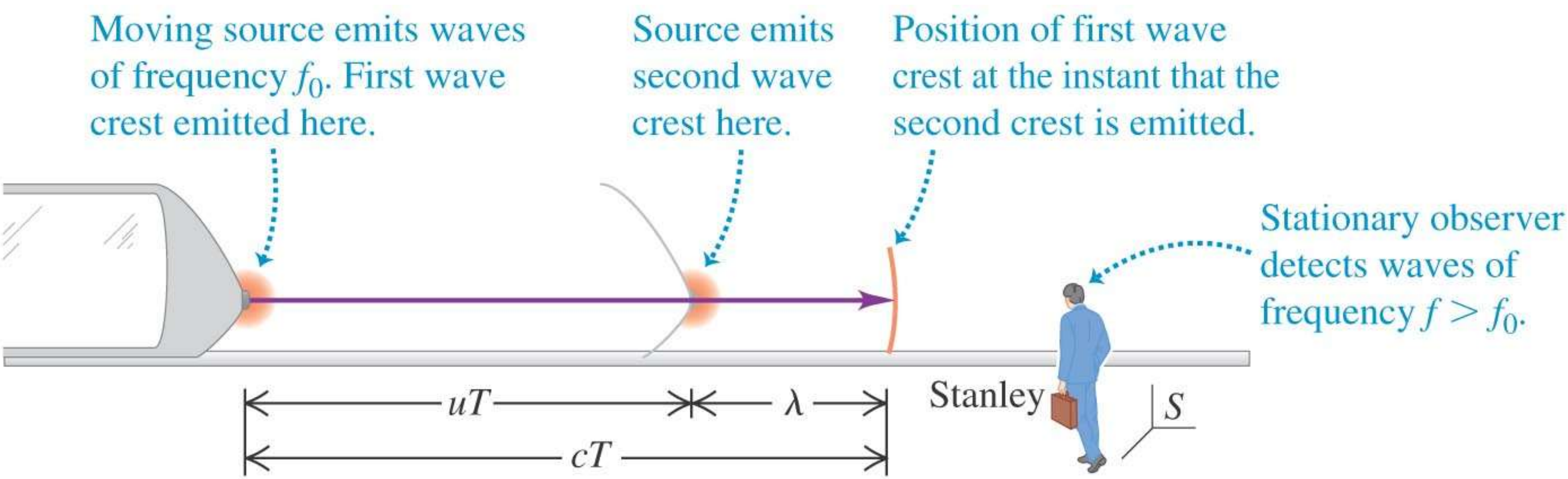
→ Both E and P will change depending on the relative S-S' velocity but $E^2 - c^2 p^2$ will not.

Note:
For particles *at rest*, $p = 0$, this expression gives $E = mc^2$ which is the rest mass energy as previously.

For *photons* with no mass, $E=pc$, which can also be shown from Maxwell's Equations.

Relativistic Doppler Effect

Statement of the problem: A source of light is *moving* at constant speed u toward a stationary observer (Stanley). The source emits EM waves with $f_0 = 1/T_0$ in its *rest frame*. What is the frequency measured by Stanley?



Relativistic Doppler Effect

$$f = \sqrt{\frac{c+u}{c-u}}f_0$$

Doppler Shift for an approaching source

higher freq → blue shifted

$$f = \sqrt{\frac{c-u}{c+u}}f_0$$

Doppler Shift for a receding source

lower freq → red shifted