2. (25 pts)
The Enterprise with a proper length of 100m is moving with respect to the earth at a speed of 4/5c. During its journey away from earth, a shuttle was launched from the back end of the Enterprise. With respect to the crew on the Enterprise, the shuttle travels at a speed of 3/5c in the same direction as the Enterprise. a) How fast is the shuttle moving with respect to an observer on earth? b) What is the length of the Enterprise as observed by the earth's observer? c) What is the length of the Enterprise as observed by the person in the shuttle? d) How much time does it take for the shuttle to reach the front end of the Enterprise according to the shuttle's observer; e) according to the Enterprise's crew; f) according to the earth's observer? [Hint: identify the correct proper time for the two events.]

a) $u = \dfrac{u'+v}{1+\dfrac{u'v}{c^2}} = \dfrac{4/5c+3/5c}{1+\dfrac{12}{25}} = \dfrac{35}{37}c = 0.946c$

The proper length L_0=100m is measured by the crew of the Enterprise who are at rest with the ship. The lengths measured by observers either on earth or on the shuttle are not proper.

b) $L_{earth} = L_0/\gamma = 100m\sqrt{1-\left(\dfrac{4}{5}\right)^2} = 60.0m$

a) $L_{shuttle} = L_0/\gamma = 100m\sqrt{1-\left(\dfrac{3}{5}\right)^2} = 80.0m$

b) The launching of the shuttle and the arrival of the shuttle are the two events defining the relevant time interval. Since the observer on the shuttle is stationary with these two events, the time interval measured by him/her is the proper time.

$$t_0 = \frac{L_{shuttle}}{3/5c} = \frac{80.0m}{0.6(3.0\times10^8 m/s)} = 4.44\times10^{-7}s$$

[with respect to Enterprise, the shuttle moves at 3/5c.]

e and f) Now, we can use the time dilation formula to get the intervals measured by the other two observers.

$$\Delta t_{Enterprise} = \frac{\Delta t_0}{\sqrt{1-\left(\dfrac{3}{5}\right)^2}} = \frac{4.44\times10^{-7}s}{\sqrt{0.64}} = 5.55\times10^{-7}s$$

$$\Delta t_{earth} = \frac{\Delta t_0}{\sqrt{1-0.946^2}} = \frac{4.44\times10^{-7}s}{\sqrt{0.105084}} = 13.7\times10^{-7}s$$

Alternative method: In the earth's frame, the Enterprise is moving also so that the shuttle needs to cover the length of the Enterprise measured in this frame as well as the distance traveled by the (front-end) of the Enterprise during this time.

$$0.946c\Delta t_{earth} = 60.0m + \frac{4}{5}c\Delta t_{earth}$$

$$\Delta t_{earth} = \frac{60.0m}{(0.946c - 0.8c)} = 13.7\times 10^{-7}\,s$$

3. (25 pts)
An elementary particle K^o meson is initially at rest with respect to the laboratory frame. It then decays into a pion π^o moving forward (in the direction the K^o meson was moving) and another pion π^o moving backward. The mass of a K^o is 498 MeV/c^2, and the mass of each pion π^o is 135 MeV/c^2. Using conservation of relativistic momentum and energy, determine the momentum and the total energy of the two emitted pions from the decay process.

BEFORE the decay, the meson is at rest so that the total energy of the system is $E_i = Mc^2$ where $M = 498 MeV/c^2$ is the rest mass of the meson and the initial momentum P_i is zero.

AFTER the decay, the total energy of the two pions is given by: $E_f = \gamma_1 mc^2 + \gamma_2 mc^2$ where $m = 135 MeV/c^2$ is the mass for the pion and $\gamma_1 = 1\big/\sqrt{1 - {v_1}^2/c^2}$ and $\gamma_2 = 1\big/\sqrt{1 - {v_2}^2/c^2}$ are the their gamma factors.

The final momentum is the sum of the momenta from the two pions:
$P_f = P_1 + P_2 = \gamma_1 m v_1 + \gamma_2 m v_2$

Then, from conservation of energy and momentum, we have following two equations:

$$E_i = E_f \Rightarrow Mc^2 = \gamma_1 mc^2 + \gamma_2 mc^2 \qquad (1)$$

$$P_i = P_f \Rightarrow 0 = \gamma_1 m v_1 + \gamma_2 m v_2 \qquad (2)$$

$$\frac{v_1}{\sqrt{1 - v_1^2/c^2}} = -\frac{v_2}{\sqrt{1 - v_2^2/c^2}}$$

From Eq. (2), we have $\frac{v_1^2}{1 - v_1^2/c^2} = \frac{v_2^2}{1 - v_2^2/c^2}$

$$v_1^2(1 - v_2^2/c^2) = v_2^2(1 - v_1^2/c^2)$$

$$v_1 = -v_2$$

and this gives $\gamma_1 = \gamma_2 = \gamma$. Substitute this back into Eq. 1, we can solve for the numerical value for γ,

$$\gamma = \frac{Mc^2}{2mc^2} = \frac{M}{2m} = \frac{498 MeV/c^2}{2 \times 135 MeV/c^2} = 1.844 .$$

Then, we can solve for the speed of the pion,

$$1 - v^2/c^2 = 1/\gamma^2$$

$$\frac{v}{c} = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/1.844^2} = 0.84$$

This give pion #1 going forward with velocity +0.84 c and pion #2 going backward with velocity -0.84c.

The momentum for pion #1 and #2 are then

$$\begin{aligned} P_1 &= \gamma m v_1 = 1.844 \cdot 135 MeV / c^2 \cdot 0.84c \\ &= 209 MeV / c \\ P_2 &= \gamma m v_2 = 1.844 \cdot 135 MeV / c^2 \cdot (-0.84c) \\ &= -209 MeV / c \end{aligned}$$

The energy for pion # 1 and #2 are:

$$\begin{aligned} E_1 = E_2 &= \gamma m c^2 = 1.844 \cdot 135 MeV / c^2 \cdot c^2 \\ &= 249 MeV \end{aligned}$$

4. (25 pts)
In a photoelectric experiment with a sodium metal surface, we find a stopping potential of 1.97 V for a light with wavelength ($\lambda = 300.0nm$) and of 0.94 V for a different light with wavelength ($\lambda = 400.0nm$). Using only this information, determine i) an experimental estimate for the Planck's constant, ii) the work function of the sodium metal surface, and iii) the cutoff frequency for this sodium surface. [express energy in units of eV]

From the two wavelengths, we have the following two equations:

$$1.97eV = hc/\lambda_1 - \phi$$

$$0.94eV = hc/\lambda_2 - \phi$$

i) Subtracting the two equations, we arrive at a equation for h,

$$1.03eV = hc/\lambda_1 - hc/\lambda_2$$

$$h = \frac{1.03eV}{c}(1/\lambda_1 - 1/\lambda_2)^{-1}$$

$$= \frac{1.03(1.602\times10^{-19}J)}{3\times10^8 m/s}(1/3 - 1/4)^{-1}(10^{-7}m)$$

$$= 6.60\times10^{-34} J\cdot s$$

This agrees with the actual Planck's constant to the second significant figure.

ii) Substituting this value for the Planck's constant back into either of the two equations, we have,

$$\phi = hc/\lambda_1 - 1.97eV$$

$$= 6.60\times10^{-34} J\cdot s(\frac{1eV}{1.602\times10^{-19}J})(3\times10^8 m/s)(\frac{1}{3\times10^{-7}m}) - 1.97eV$$

$$= 4.12eV - 1.97eV$$

$$= 2.15eV$$

iii) The cutoff frequency is

$$hf_c = \phi$$

$$f_c = 2.15eV(1.602\times10^{-19}J/1eV)/6.60\times10^{-34}J\cdot s$$

$$= 5.22\times10^{14}Hz$$

This frequency corresponds to a wavelength of 575 nm.

5. (25 pts)

An electron is trapped inside a rigid box of length L=0.250nm. a) If the electron is initially in the *second* excited state, what is the wavelength of the emitted photon if the electron jumps to the *ground* state? b) The wavefunction for the electron in its *first excited* state is given by $\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$. What is the probability of finding the electron in the middle region of the rigid box, $\frac{L}{4}\le x\le\frac{3L}{4}$. c) Sketch the probability density function for this first excited state. What is (are) the most probable location(s) in finding the electron in its first excited state? (The mass of an electron is 0.511 MeV/c^2 or 9.11x10^{-31}kg.) $[\sin^2 x=\frac{1}{2}(1-\cos 2x)]$

a) The energy levels for an electron inside a rigid box is given by

$$E_n=\frac{h^2}{8mL^2}n^2,\quad n=1,2,3,\cdots$$

For a transition from n=3 (2$^{\text{nd}}$ excited state) to n=1 (ground state), the emitted photon will have energy given by,

$$hf=\frac{h^2}{8mL^2}\left(3^2-1\right)$$

$$\frac{1}{\lambda}=\frac{8hc}{8mL^2c^2}=\frac{1240eV\cdot nm}{(0.511\times10^6 eV/c^2)(0.25nm)^2c^2}=0.03883nm^{-1}$$

$$\lambda=25.8nm$$

b) The probability of finding the particle in the desired region is given by

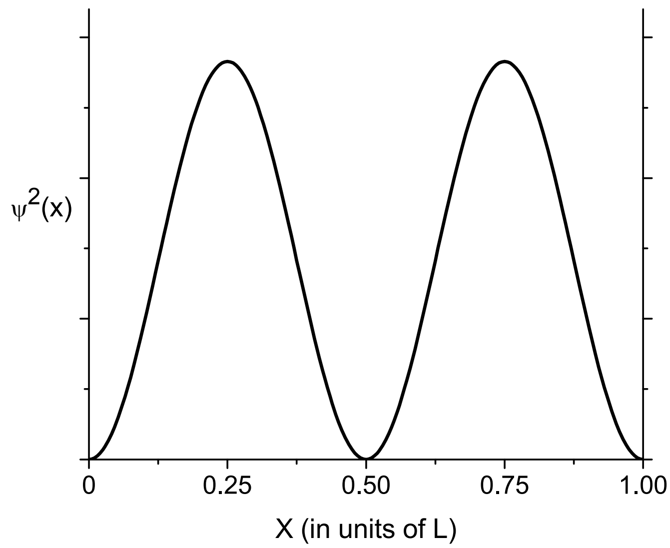
$$P=\int_{L/4}^{3L/4}\left|\psi(x)\right|^2dx=\frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{2\pi x}{L}\right)dx$$

$$=\frac{1}{L}\int_{L/4}^{3L/4}\left[1-\cos\left(\frac{4\pi x}{L}\right)\right]dx$$

$$=\frac{1}{L}\left[\frac{L}{2}+\sin\left(\frac{4\pi x}{L}\right)\Bigg|_{L/4}^{3L/4}\right]$$

$$=\frac{1}{L}\left[\frac{L}{2}+\left(\sin\left(3\pi\right)-\sin\left(\pi\right)\right)\right]=1/2$$

c) A graph of $|\psi(x)|^2$ is given below



So, the most probable locations in finding the electron are at the two peaks located at $x = L/4$ and $3L/4$.