2. (25 pts) When a high energy cosmic ray enters the top of Earth's atmosphere, 4.00km above sea level, a muon is created. The muon has a total relativistic energy E of 954 MeV (with respect to an observer in the Earth's frame) and is traveling vertically downward. In the muon's rest frame, the muon has a life-time of 1.56 μs. a) Calculate the speed of the muon in units of c. b) Will this muon reach the surface of the Earth? c) If the answer is yes, how much time does it take for the muon to reach the surface of the Earth in the muon's rest frame? d) If the answer to b) is no, at what altitude above sea level, according to an observer on the surface of the Earth, does the muon vanish when it decays? (The rest mass of the muon is 106 MeV/c^2 or $1.89\times10^{-28}\,kg$.)

Solution:

a) The total relativistic energy of the pion is $E=\gamma mc^2$.

So, $\gamma=\frac{E}{mc^2}=\frac{954MeV}{(106MeV/c^2)\cdot c^2}=9$.

From the definition of the Lorentz factor, we have

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow 1-\frac{v^2}{c^2}=\frac{1}{\gamma^2} \rightarrow \frac{v^2}{c^2}=1-\frac{1}{\gamma^2} \rightarrow v=\sqrt{1-\frac{1}{\gamma^2}}\,c=\sqrt{1-\frac{1}{81}}\,c$$

This gives $v=0.994c$.

b) Then, according to the muon, the distance that it needs to travel is Lorentz contracted,
$L=L_0/\gamma=4km/9=0.444km=444m$.

But, with its limited life-time, the maximum distance that it can travel is
$d'=\tau_0 v=1.56\times10^{-6}s(0.994)(3\times10^8m/s)=465m$

Since $d' > L$, this muon will reach the surface of the Earth.

c) In the muon's rest frame, it has a contracted distance of 444m to travel and the time of travel t_0 according to the muon will be,

$$t_0=L/v=\frac{4000m}{9}\left(\frac{1}{0.9938\left(3\times10^8m/s\right)}\right)=1.49\times10^{-6}s=1.49\mu s$$

So, in the muon's rest frame, it will reach the surface 0.07 μs before it decays.

d) It will reach the Earth's surface.

3. (25 pts)
An elementary particle K^o meson is initially at rest with respect to the laboratory frame. It then decays into a pion π^o moving forward (in the direction the K^o meson was moving) and another pion π^o moving backward. The mass of a K^o is 498 MeV/c^2, and the mass of each pion π^o is 135 MeV/c^2. Using conservation of relativistic momentum and energy, determine the momentum and the total energy of the two emitted pions from the decay process.

BEFORE the decay, the meson is at rest so that the total energy of the system is $E_i = Mc^2$ where $M = 498 MeV/c^2$ is the rest mass of the meson and the initial momentum P_i is zero.

AFTER the decay, the total energy of the two pions is given by: $E_f = \gamma_1 mc^2 + \gamma_2 mc^2$ where $m = 135 MeV/c^2$ is the mass for the pion and $\gamma_1 = 1\big/\sqrt{1 - v_1^{\,2}/c^2}$ and $\gamma_2 = 1\big/\sqrt{1 - v_2^{\,2}/c^2}$ are the their gamma factors.

The final momentum is the sum of the momenta from the two pions:
$P_f = P_1 + P_2 = \gamma_1 m v_1 + \gamma_2 m v_2$

Then, from conservation of energy and momentum, we have following two equations:
$E_i = E_f \Rightarrow Mc^2 = \gamma_1 mc^2 + \gamma_2 mc^2 \qquad (1)$
$P_i = P_f \Rightarrow 0 = \gamma_1 m v_1 + \gamma_2 m v_2 \qquad (2)$

$$\frac{v_1}{\sqrt{1 - v_1^2/c^2}} = -\frac{v_2}{\sqrt{1 - v_2^2/c^2}}$$

From Eq. (2), we have $\dfrac{v_1^2}{1 - v_1^2/c^2} = \dfrac{v_2^2}{1 - v_2^2/c^2}$

$$v_1^2(1 - v_2^2/c^2) = v_2^2(1 - v_1^2/c^2)$$

$$v_1 = -v_2$$

and this gives $\gamma_1 = \gamma_2 = \gamma$. Substitute this back into Eq. 1, we can solve for the numerical value for γ,

$$\gamma = \frac{Mc^2}{2mc^2} = \frac{M}{2m} = \frac{498 MeV/c^2}{2 \times 135 MeV/c^2} = 1.844.$$

Then, we can solve for the speed of the pion,
$1 - v^2/c^2 = 1/\gamma^2$

$$\frac{v}{c} = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/1.844^2} = 0.84$$

This give pion #1 going forward with velocity +0.84 c and pion #2 going backward with velocity -0.84c.

The momentum for pion #1 and #2 are then

$$
\begin{aligned}
P_1 &= \gamma m v_1 = 1.844 \cdot 135 MeV / c^2 \cdot 0.84c \\
&= 209 MeV / c \\
P_2 &= \gamma m v_2 = 1.844 \cdot 135 MeV / c^2 \cdot (-0.84c) \\
&= -209 MeV / c
\end{aligned}
$$

The energy for pion # 1 and #2 are:

$$
\begin{aligned}
E_1 = E_2 &= \gamma mc^2 = 1.844 \cdot 135 MeV / c^2 \cdot c^2 \\
&= 249 MeV
\end{aligned}
$$

4. (25 pts)

In a photoelectric experiment with a sodium metal surface, we find a stopping potential of 1.97 V for a light with wavelength ($\lambda = 300.0nm$) and of 0.94 V for a different light with wavelength ($\lambda = 400.0nm$). Using only this information, determine i) an experimental estimate for the Planck's constant, ii) the work function of the sodium metal surface, and iii) the cutoff frequency for this sodium surface. [express energy in units of eV]

From the two wavelengths, we have the following two equations:

$1.97eV = hc/\lambda_1 - \phi$

$0.94eV = hc/\lambda_2 - \phi$

i) Subtracting the two equations, we arrive at a equation for h,

$$1.03eV = hc/\lambda_1 - hc/\lambda_2$$

$$h = \frac{1.03eV}{c}(1/\lambda_1 - 1/\lambda_2)^{-1}$$

$$= \frac{1.03(1.602\times10^{-19}J)}{3\times10^8 m/s}(1/3 - 1/4)^{-1}(10^{-7}m)$$

$$= 6.60\times10^{-34} J\cdot s$$

This agrees with the actual Planck's constant to the second significant figure.

ii) Substituting this value for the Planck's constant back into either of the two equations, we have,

$$\phi = hc/\lambda_1 - 1.97eV$$

$$= 6.60\times10^{-34} J\cdot s(\frac{1eV}{1.602\times10^{-19}J})(3\times10^8 m/s)(\frac{1}{3\times10^{-7}m}) - 1.97eV$$

$$= 4.12eV - 1.97eV$$

$$= 2.15eV$$

iii) The cutoff frequency is

$$hf_c = \phi$$

$$f_c = 2.15eV(1.602\times10^{-19}J/1eV)/6.60\times10^{-34}J\cdot s$$

$$= 5.22\times10^{14} Hz$$

This frequency corresponds to a wavelength of 575 nm.

5. (25 pts)

An electron is trapped inside a rigid box of length L=0.250nm. a) If the electron is initially in the *second* excited state, what is the wavelength of the emitted photon if the electron jumps to the *ground* state? b) The wavefunction for the electron in its *first excited* state is given by $\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$. What is the probability of finding the electron in the middle region of the rigid box, $\frac{L}{4}\le x\le\frac{3L}{4}$. c) Sketch the probability density function for this first excited state. What is (are) the most probable location(s) in finding the electron in its first excited state? (The mass of an electron is 0.511 MeV/c^2 or 9.11x10^{-31}kg.) [$\sin^2 x=\frac{1}{2}(1-\cos 2x)$]

a) The energy levels for an electron inside a rigid box is given by

$$E_n=\frac{h^2}{8mL^2}n^2,\quad n=1,2,3,\cdots$$

For a transition from n=3 (2nd excited state) to n=1 (ground state), the emitted photon will have energy given by,

$$hf=\frac{h^2}{8mL^2}\left(3^2-1\right)$$

$$\frac{1}{\lambda}=\frac{8hc}{8mL^2c^2}=\frac{1240eV\cdot nm}{(0.511\times10^6 eV/c^2)(0.25nm)^2c^2}=0.03883nm^{-1}$$

$$\lambda=25.8nm$$

b) The probability of finding the particle in the desired region is given by

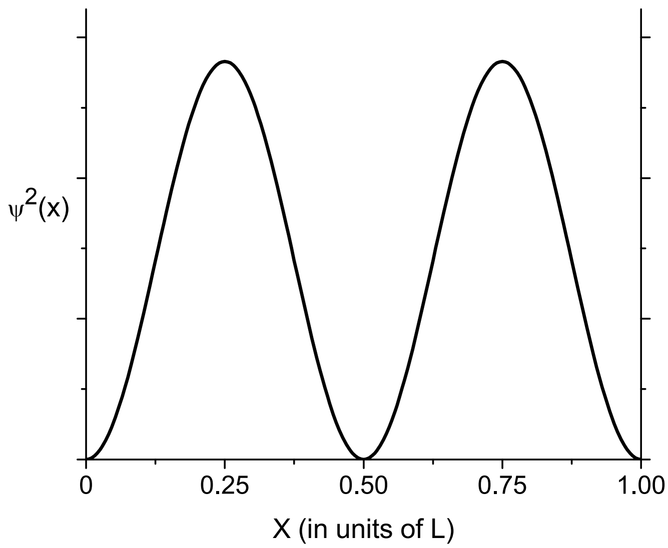
$$P=\int_{L/4}^{3L/4}\left|\psi(x)\right|^2dx=\frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{2\pi x}{L}\right)dx$$

$$=\frac{1}{L}\int_{L/4}^{3L/4}\left[1-\cos\left(\frac{4\pi x}{L}\right)\right]dx$$

$$=\frac{1}{L}\left[\frac{L}{2}+\sin\left(\frac{4\pi x}{L}\right)\Bigg|_{L/4}^{3L/4}\right]$$

$$=\frac{1}{L}\left[\frac{L}{2}+\left(\sin\left(3\pi\right)-\sin\left(\pi\right)\right)\right]=1/2$$

c) A graph of $|\psi(x)|^2$ is given below



So, the most probable locations in finding the electron are at the two peaks located at $x=L/4$ and $3L/4$.

5. What is the longest wavelength of light capable of ionizing a hydrogen atom in the ground state? What happens if the wavelength is shorter than this value? [useful constant: hc=1242 eV.nm, $1eV = 1.602\times10^{-19}J$, $h = 6.626\times10^{-34}J\cdot s$]

Solution

The energy required to ionize a hydrogen atom is 13.6 eV. The photon capable of producing this ionization must have at least this energy, if not higher. Since the photon energy is E = hf = hc/λ, it must have the wavelength of

λ = hc/E = (1.24×10^3 eV·nm)/(13.6 eV) = 91.2 nm

or smaller.

Thus, the largest wavelength of light capable of ionizing the hydrogen atom is 91.2 nm, which is UV light. The electron knocked out of the atom by this photon has zero kinetic energy. Photons with smaller wavelengths also produce the ionization but the electron removed from the atom has a kinetic energy of K = hc/λ - 13.6 eV.