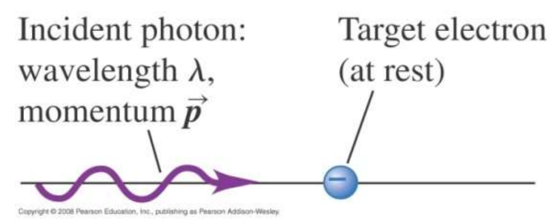
Compton Scattering

Conservation of Total Relativistic Energy:

Note: high energy collision → e's speed might be relativistic !

(before) (after)

$$\frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \gamma mc^2$$

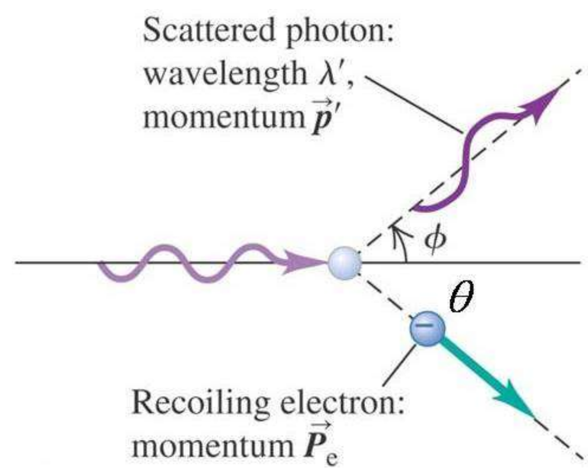


Conservation of Relativistic Momentum:

(before) (after)

$$x-dir: \quad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + \gamma mv\cos\theta$$

$$y-dir: \quad 0 = \frac{h}{\lambda'}\sin\phi - \gamma mv\sin\theta$$



(knowns → λ: given, ϕ: observation angle) 3 eqs. with 3 unknowns (θ, λ', v) !

Compton Scattering

Dividing c throughout the energy conservation equation, here are the three equations:

$$(1) \quad \frac{h}{\lambda} + mc = \frac{h}{\lambda'} + \gamma mc$$

$$(2) \quad \frac{h}{\lambda} = \frac{h}{\lambda'} \cos\phi + \gamma mv \cos\theta$$

$$(3) \quad 0 = \frac{h}{\lambda'} \sin\phi - \gamma mv \sin\theta$$

$$\vdots$$

Compton Scattering

Dividing c throughout the energy conservation equation and define the following shorthand: $g=h/\lambda$ and $g'=h/\lambda'$, the three eqs. can be written as:

$$(1)\quad g+mc=g'+\gamma mc$$

$$(2)\quad g=g'\cos\phi+\gamma mv\cos\theta$$

$$(3)\quad 0=g'\sin\phi-\gamma mv\sin\theta$$

$(1)^2$ →

$$\left(g-g'+mc\right)^2=\gamma^2m^2c^2$$

$$\left(g-g'\right)^2+2\left(g-g'\right)mc+m^2c^2=\gamma^2m^2c^2$$

$$\left(g-g'\right)^2+2\left(g-g'\right)mc=\left(\gamma^2-1\right)m^2c^2=\left(\frac{1}{1-v^2/c^2}-1\right)m^2c^2$$

$$\left(g-g'\right)^2+2\left(g-g'\right)mc=\left(\frac{\cancel{1}-\cancel{1}+v^2/c^2}{1-v^2/c^2}\right)m^2c^2=\frac{m^2v^2}{1-v^2/c^2}$$

Compton Scattering

$(2)^2 + (3)^2 \rightarrow$

$$\begin{aligned} (g - g'\cos\phi)^2 &= \gamma^2 m^2 v^2 \cos^2\theta \\ \oplus \quad g'^2 \sin^2\phi &= \gamma^2 m^2 v^2 \sin^2\theta \end{aligned}$$

$$\boxed{g^2 - 2gg'\cos\phi + g'^2\left(\cos^2\phi\right.} \left. + \sin^2\phi\right) = \gamma^2 m^2 v^2 \left(\cos^2\theta + \sin^2\theta\right)$$

$$g^2 - 2gg'\cos\phi + g'^2(1) = \frac{m^2 v^2}{1 - v^2/c^2}(1)$$

Notice that the RHS is the same as the one from the energy conservation equation. Equating them gives:

$$\underbrace{(g - g')^2 + 2(g - g')mc}_{(1)^2} = \underbrace{g^2 - 2gg'\cos\phi + g'^2}_{(2)^2 + (3)^2}$$

$$\cancel{g^2} - \cancel{2}gg' + \cancel{g'^2} + \cancel{2}(g - g')mc = \cancel{g^2} - \cancel{2}gg'\cos\phi + \cancel{g'^2}$$

$$(g - g')mc = gg'(1 - \cos\phi)$$

Compton Scattering

Substituting the shorthand *g*=*h*/*λ* and *g*'=*h*/*λ*' back:

$$\left(\frac{\cancel{h}}{\lambda}-\frac{\cancel{h}}{\lambda'}\right)mc=\frac{h^{\cancel{2}}}{\lambda\lambda'}(1-\cos\phi)$$

$$\lambda\lambda'\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\lambda'-\lambda=\frac{h}{mc}(1-\cos\phi)$$

(Compton Shift Equation)

The factor *h*/*mc* has the units of length and

$$\lambda_C=\frac{h}{mc}=0.002426nm$$

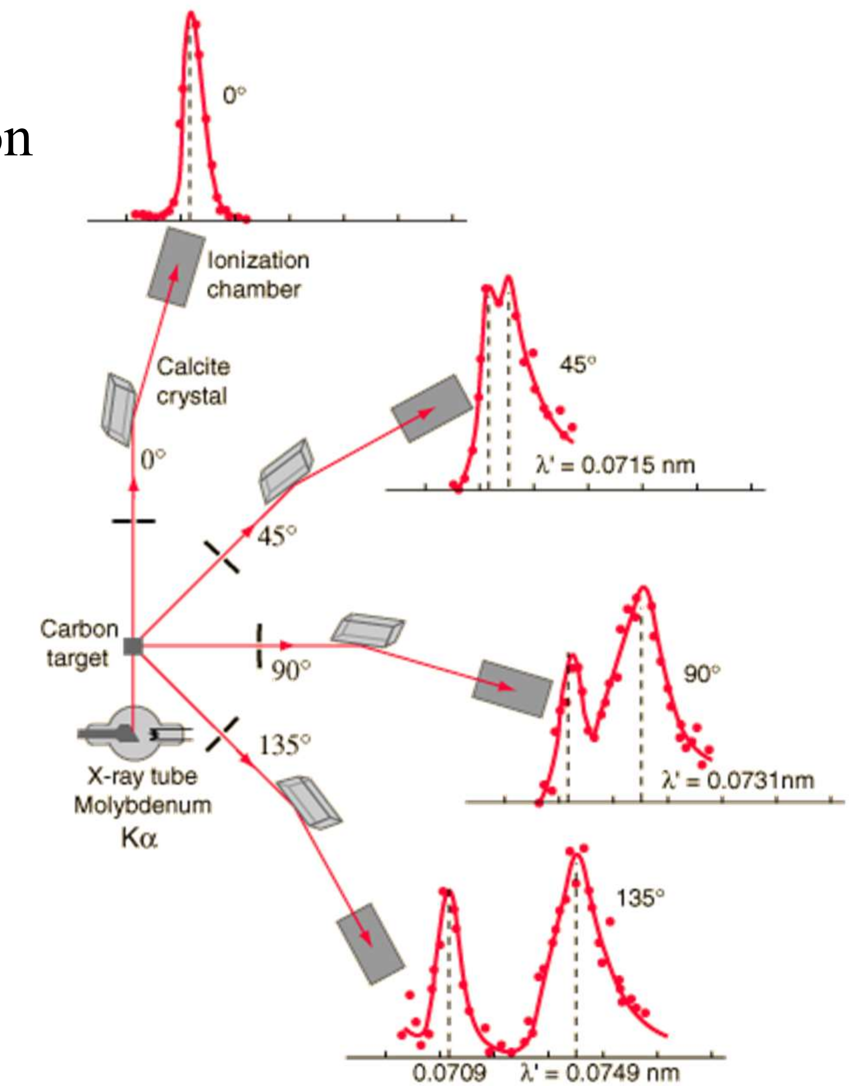
is called the **Compton Wavelength**.

Compton Scattering

Most importantly, the Compton Shift equation

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

gives the exact prediction for the observed wavelength shift in the Compton scattering experiment.

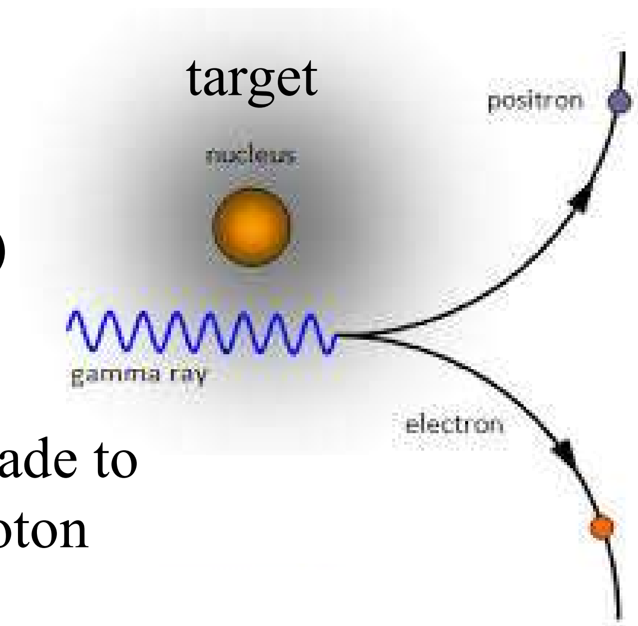


The particle description for photon is correct !

Gamma Rays and Pair Production

What happens when an even more energetic light (photon) interacts with matter?

γ - rays are high energy EM radiation with shorter λ and higher f even than X - rays



Patrick Blackett and Giuseppe Occhialini (1933) found that…

→ When a sufficiently high energy γ - rays is made to collide with a nucleus within a target, the photon will disappear AND a pair of electron e^- and **positron** e^+ (the anti-particle of e^-) can be created.

This process is called **Pair Production**.

Gamma Rays and Pair Production

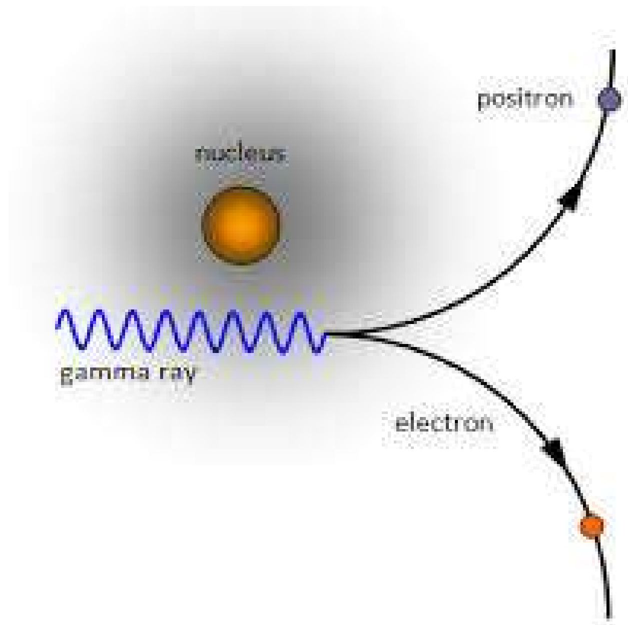
Physical requirements:

- Energy of γ must be sufficient high:

$$E_{\min} \geq 2m_e c^2 = 2(0.511MeV) = 1.022MeV$$

$$f_\gamma(\min) = \frac{E_{\min}}{h} = 247.3EHz\,(exa - E\,10^{18})$$

$$\lambda_\gamma(\max) = 1.213\,pm\,(pico - p\,10^{-12})$$



In addition to energy conservation,

- Charge is conserved:

$$(-e)+(+e)=0$$

γ has no charge

- Momentum is conserved through the recoil of the nucleus.

Again, explanation needs light as particles !

Correspondence Principle

So, when does light acts *classically* and *quantum mechanically*?

As we have seen, the basic energy scale for light (a photon) is $\Delta E = hf$

When light interacts with matter, the typical interaction energy scale E_{int} as compares to a single photon energy ΔE determines whether light acts as a wave (CM description) or a particle (QM description):

if f is *small* (λ is long), $\Delta E = hf \ll E_{\text{int}}$ then light appears as a continuum
EM → CM (light as a wave)

If f is *large* (λ is short), $\Delta E = hf \approx E_{\text{int}}$ then light appears as discrete packets
EM → QM (light as particles)

Correspondence Principle

Radio waves (low freq): behaves like *classical* waves with diffraction and interference easily observable.

Visible lights (mid freq):
have both wave (classical) & particle (quantum) behaviors.

diffraction

photoelectric effect

X-Rays (high freq): mostly particle (quantum) behaviors.

X-Rays productions and Compton Scattering

γ-Rays (very high freq): particle (quantum) behaviors.

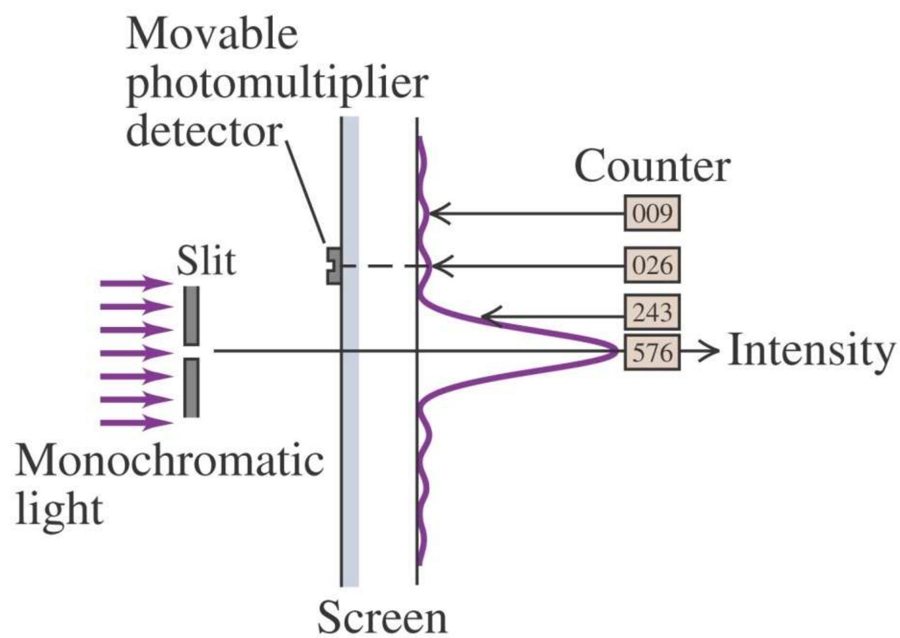
Pair Productions and e^--e^+ Pair Annihilations (reverse of PP)

Wave-Particle Duality

The Principle of Complementarity: First stated by Niels Bohr in 1928

The wave and the particle descriptions are complementary. We need both descriptions to complete our model of nature, but we never need to use both descriptions at the same time to describe a single part of an occurrence.

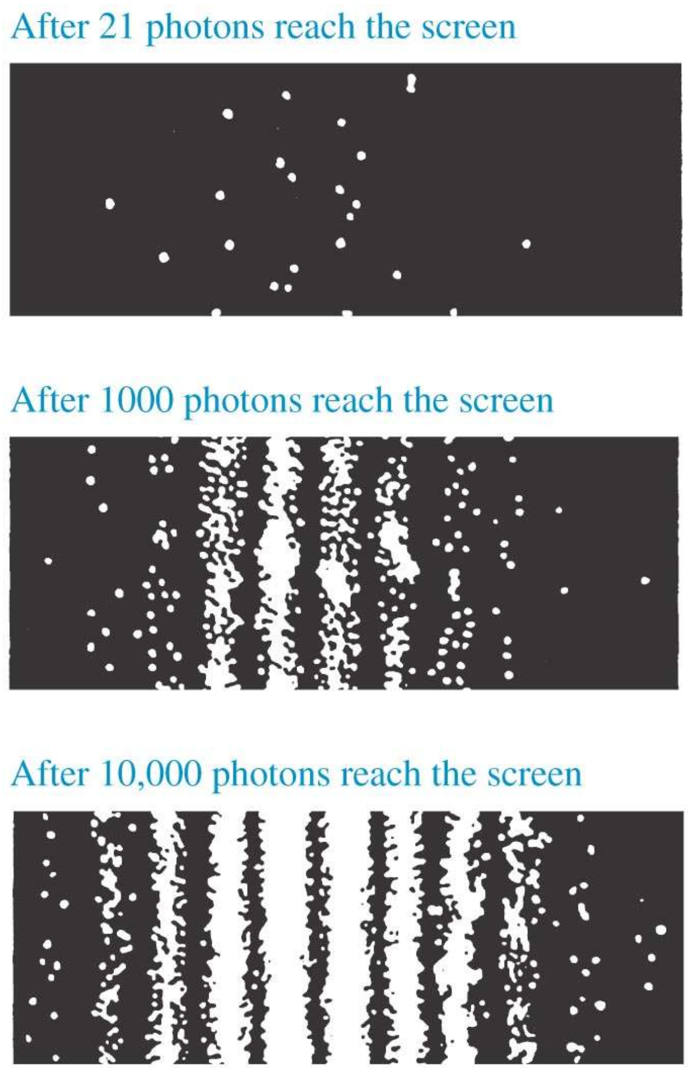
Modify our light diffraction experiment slightly: use a photomultiplier tube to measure the number of photons arriving at a given location.



Wave-Particle Duality

Low intensity result:

(one or a few photons passing the slits at one time.)

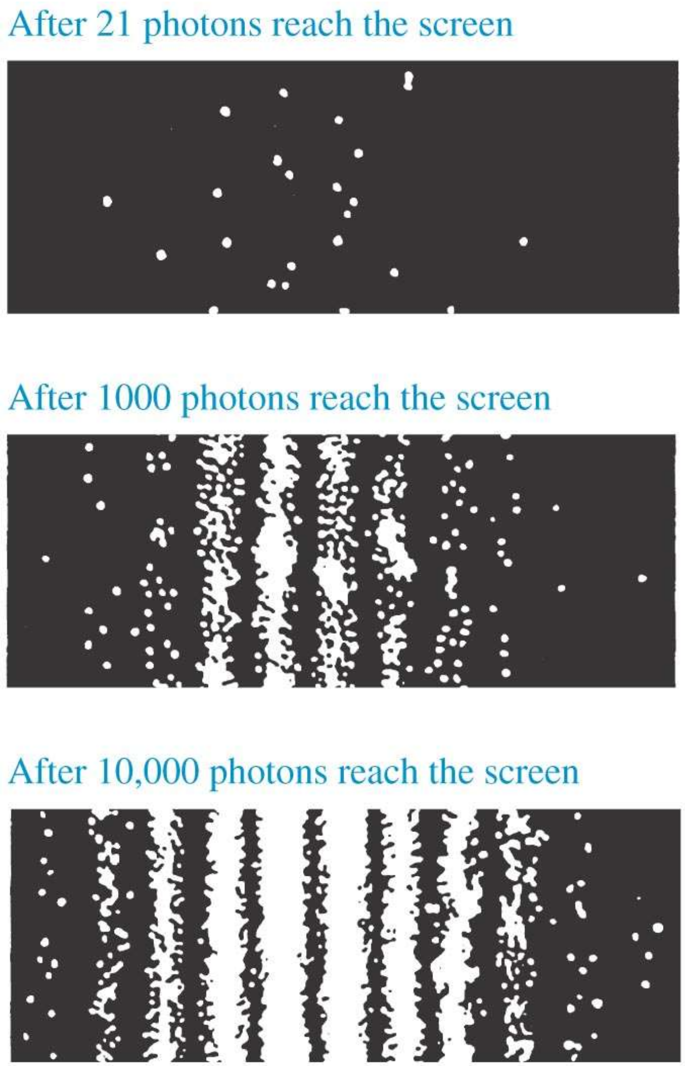


Photon detections by photomultiplier tube must be done over time.

The arrival of a photon at a particular location will be probabilistic.

And, results are statistical in nature.

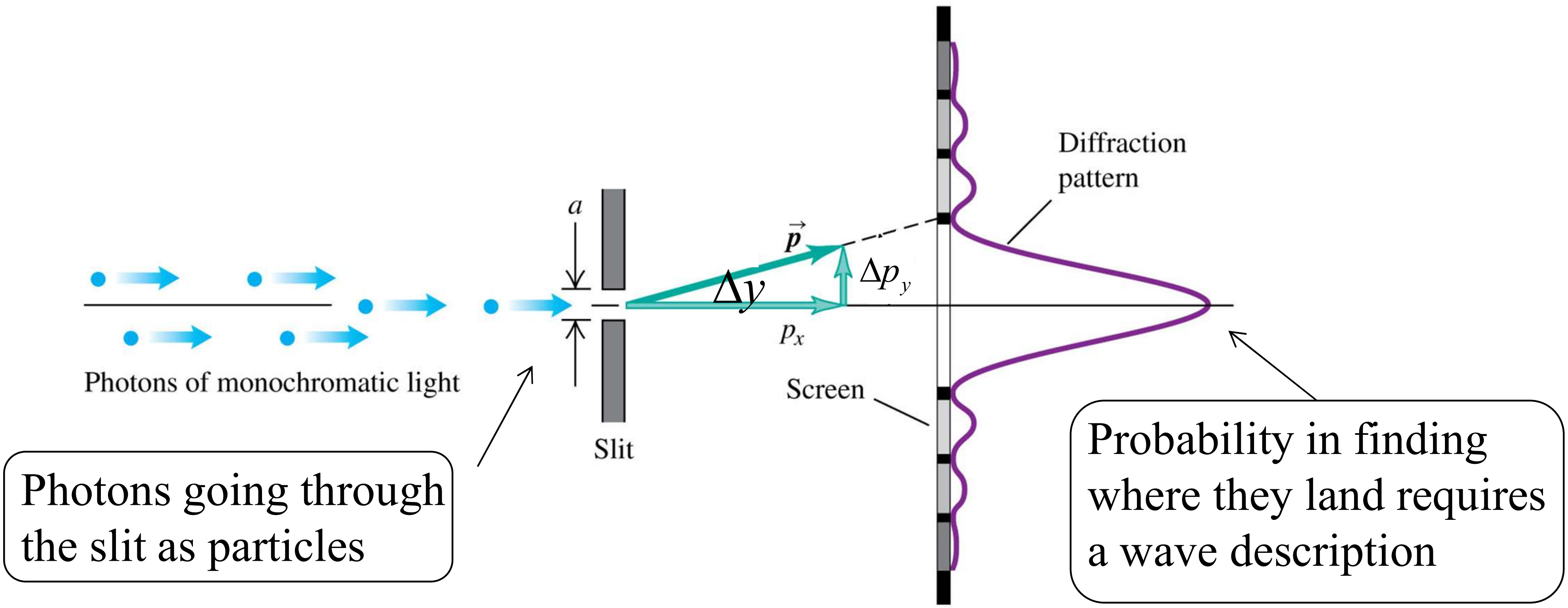
Wave-Particle Duality



The detection of a single photon in the photomultiplier is a ***particle*** measurement.

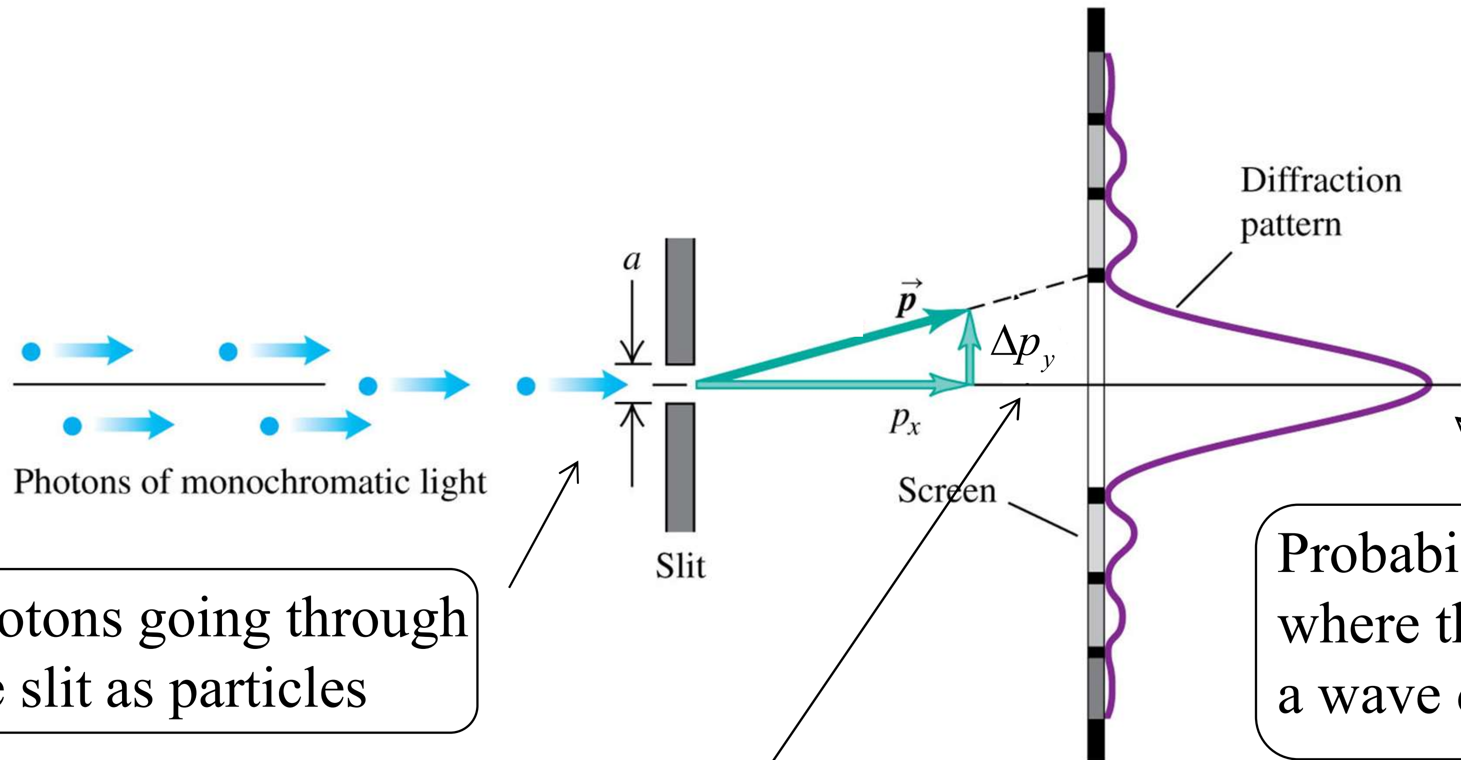
But, the distribution pattern of photons arriving at the screen requires a ***wave*** description.

Heisenberg's Uncertainty Principle



For photons going through the slit with width a, one can say that the y-position of the photons will have an uncertain of Δy

Heisenberg's Uncertainty Principle



Photons going through the slit as particles

Probability in finding where they land requires a wave description

Also, the "spreading-out" of the photons implies that the photons reaching the screen must have a small *uncertainty* Δp_y in the vertical ($\hat{\mathbf{y}}$) direction of its momentum as it exits the slit.

Heisenberg Uncertainty Principle again

The Heisenberg Uncertainty Principle is not a statement on the accuracy of the experimental instruments. It is a statement on the fundamental *limitations* in *making* measurements !

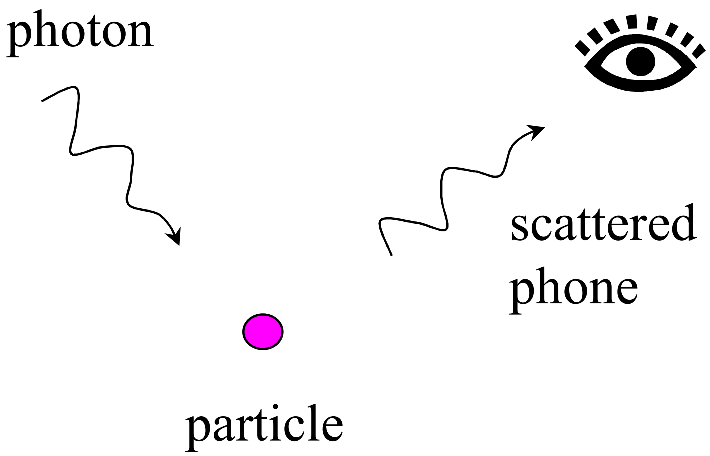
To understand this, there are two important factors:

1. The wave-particle duality
2. The unavoidable interactions between the observer and the object being observed

Uncertainty Principle: Conceptual Example

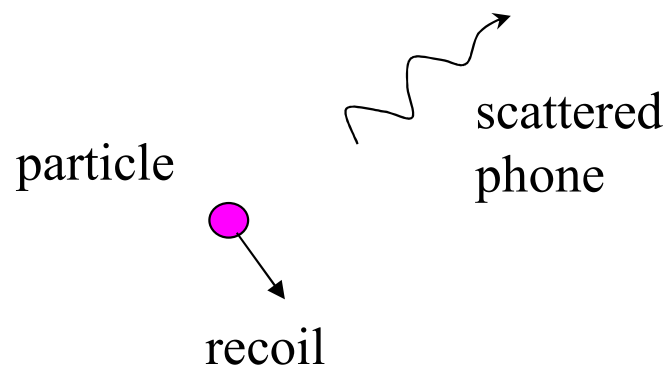
Suppose we want to measure the position of a particle (originally at rest) by a laser light.

The measurement is accomplished by the scattering of photons off this particle (shining a light) :



Uncertainty Principle: Conceptual Example

But, photons carry momentum and the particle after scattering will recoil !



While the scattered photon gives us a precise position of the particle, the photon will also inevitably impart momentum onto the particle.

Uncertainty Principle: Conceptual Example

If the measurement is made by a photon with wavelength λ, then *at best*,

we can expect our position measurement to be accurate up to an uncertainty,

$$\Delta y \sim \lambda$$

But, the photon will transfer a part of its momentum to the particle along the measurement direction y. The amount transfer is then given by,

$$\Delta p_y \sim \frac{h}{\lambda}$$

Combining these two expressions gives: $\Delta y \Delta p_y \sim h$

NOTE: By improving Δy by using a *smaller* λ will inadvertently result in a larger uncertainty in $\Delta p_y \sim h/\lambda$ since Δp_y is inverse proportional to λ !.

Heisenberg's Uncertainty Principle

We have used a heuristic argument in getting to previous inequality. A more precise inequality can be derivate from full QM calculations, and it should read

$$\Delta y \Delta p_y \geq \frac{\hbar}{2} \qquad \text{where } \hbar = h/2\pi$$

This is called the **Heisenberg's Uncertainty Principle**.

Uncertainty Relation for ΔE & Δt

Our discussion is not specific to the $\hat{\mathbf{y}}$ direction. In general, we will have similar uncertainty relations for all three spatial directions, i.e.,

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}$$

And, additionally, there is an uncertainty relation for ΔE and Δt :

$$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad \text{similar to} \qquad \Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Uncertainty Relation for ΔE & Δt

In nature, space-time coordinates are linked to its dynamical counterparts as **conjugate** variable pairs in physics.

$$(p_x, x) \quad (p_y, y) \quad (p_z, z) \quad \text{and} \quad (E, t)$$

And, most importantly, the Heisenberg's Uncertainty Principle enforces an *inverse* proportional relation on the two *conjugate* pairs of dynamics variables:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

By decreasing the uncertainty in one of the variables (x or t), its corresponding **conjugate** variable $(p_x \text{ or } E)$ must increase accordingly !

But, there are **no** restrictions for *unconjugated* variables: $\Delta x \Delta p_y$ *or* $\Delta x \Delta y$, *etc.*

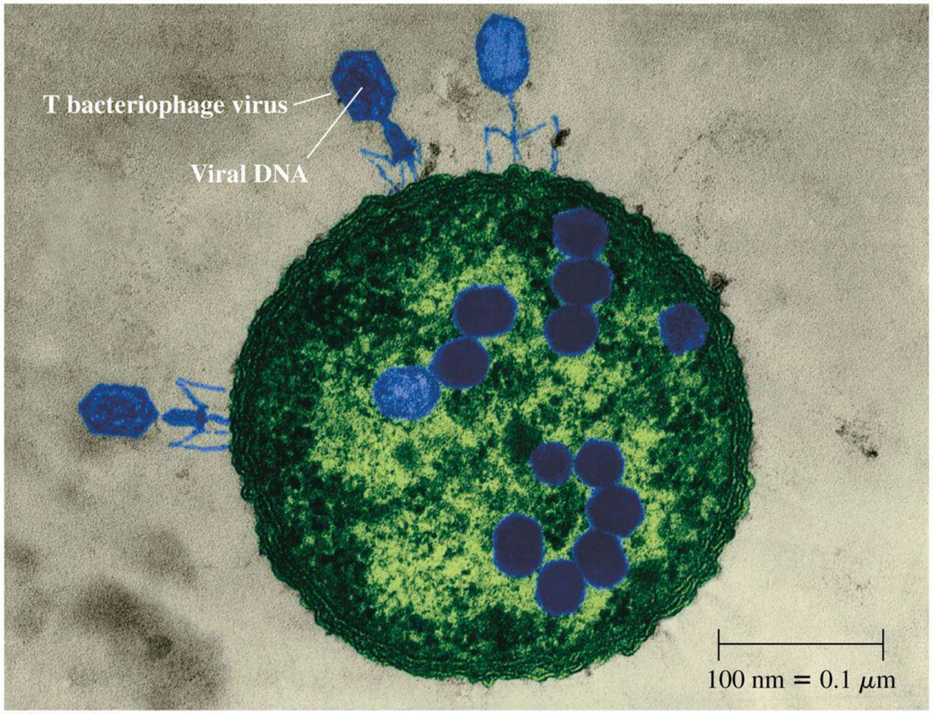
PHYS 262

George Mason University

Prof. Paul So

Chapter 39: Particles Behaving as Waves

- Matter Waves
- Atomic Line-Spectra and Energy Levels
- Bohr's Model of H-atom
- The Laser
- Continuous Spectra & Blackbody Radiation



Matter Waves

As we have seen, light has a *duality* of being a wave and a particle.

By a *symmetry* argument, de Broglie in 1924 proposed that *all form of matter* should also posses this duality.

Recall for photons, we have: $\lambda = \dfrac{h}{p}$

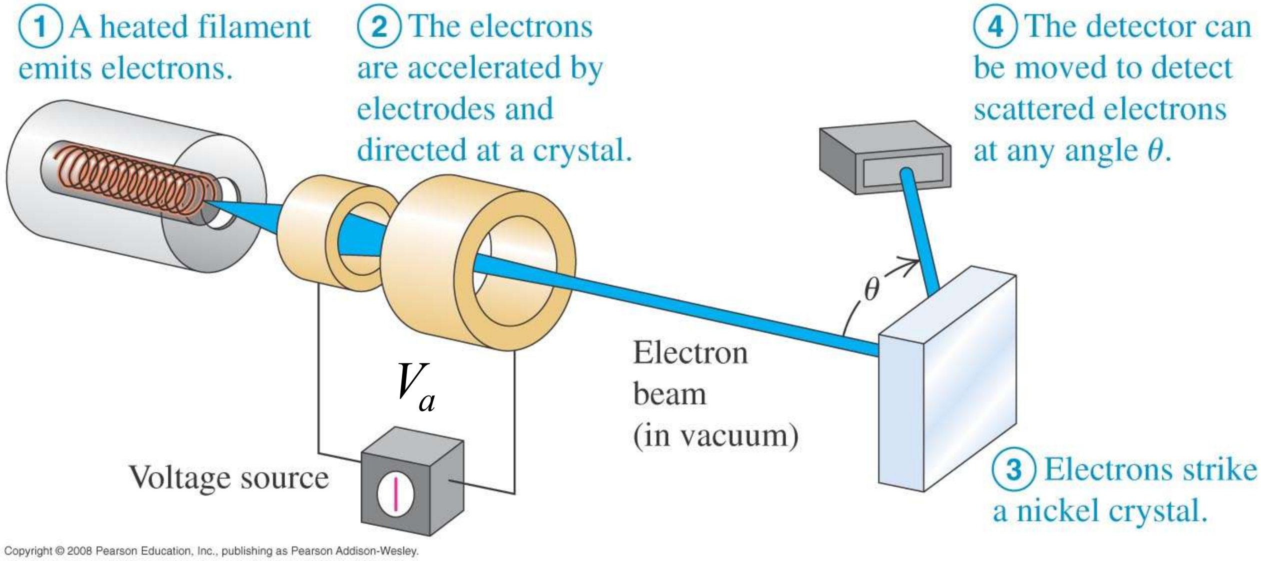
For a massive particle with momentum $p = mv$ (or γmv) and total energy E, de Broglie proposed:

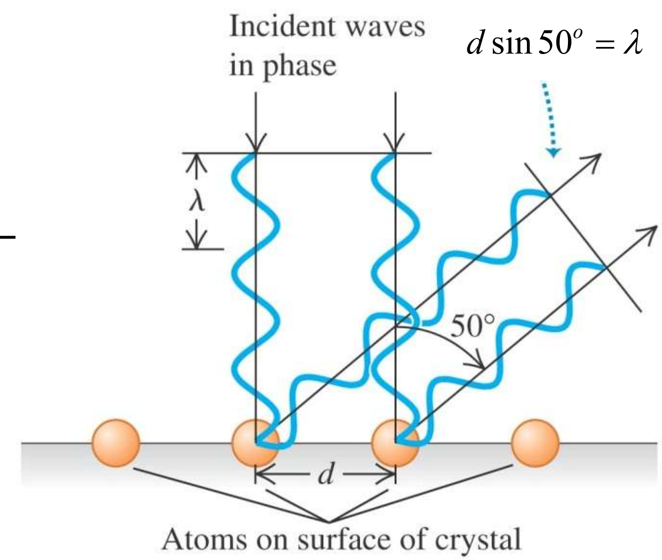
$$\lambda = \frac{h}{p} = \frac{h}{mv} \quad \left(= \frac{h}{\gamma mv} \right) \qquad f = \frac{E}{h}$$

(de Broglie wavelength)

Electron Diffraction

Clinton Davisson and Lester Germer (1927)

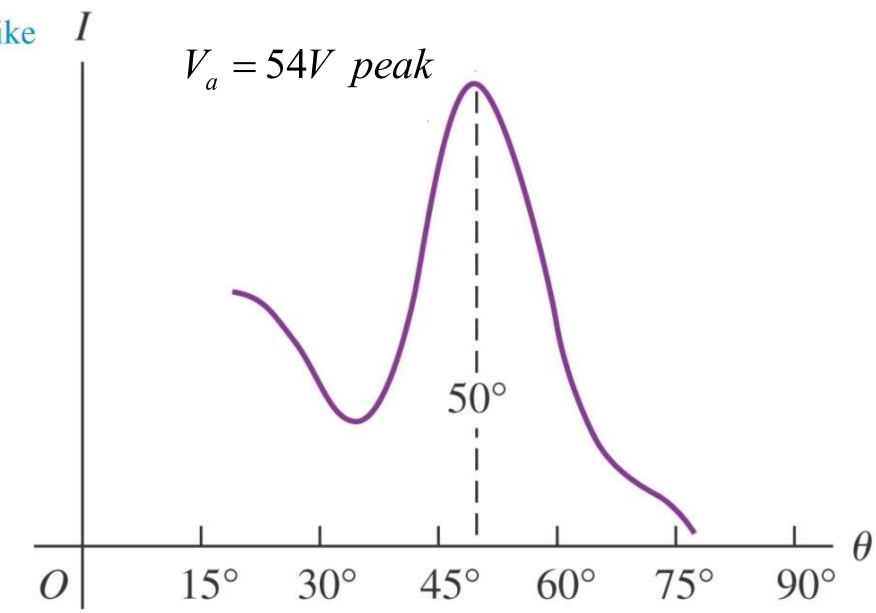




de Brogie's wavelength of e,

$$eV_a = p^2/2m$$

$$\lambda = h/p = h/\sqrt{2meV_a}$$



Nickel atoms in crystal acts as a diffraction grading,

Constructive interference is expected at:

$$d\sin\theta = m\lambda, \quad m = 1, 2, \cdots$$

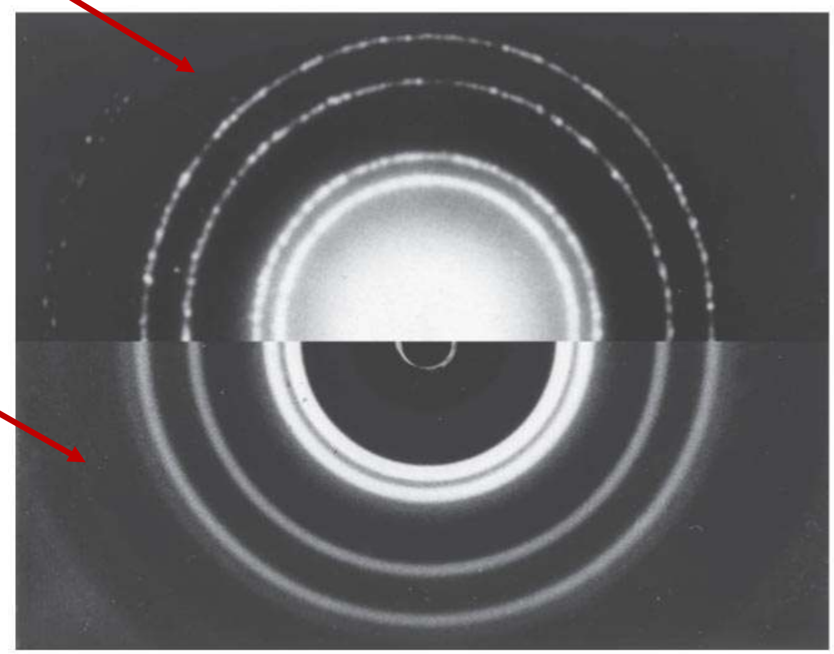
Diffraction of *e* off Aluminum Foil

In 1928, G.P. Thomson (son of J.J. Thomson) performed another demonstrative experiment in showing electrons can act as a wave and diffract from a polycrystalline aluminum foil.

Debye and Sherrer did a similar X-Ray diffraction experiment done a few years earlier using a similar set up.

G.P. Thomson's electron diffraction experiment produced a qualitatively similar result as Debye & Sherrer's X-Ray diffraction experiment.

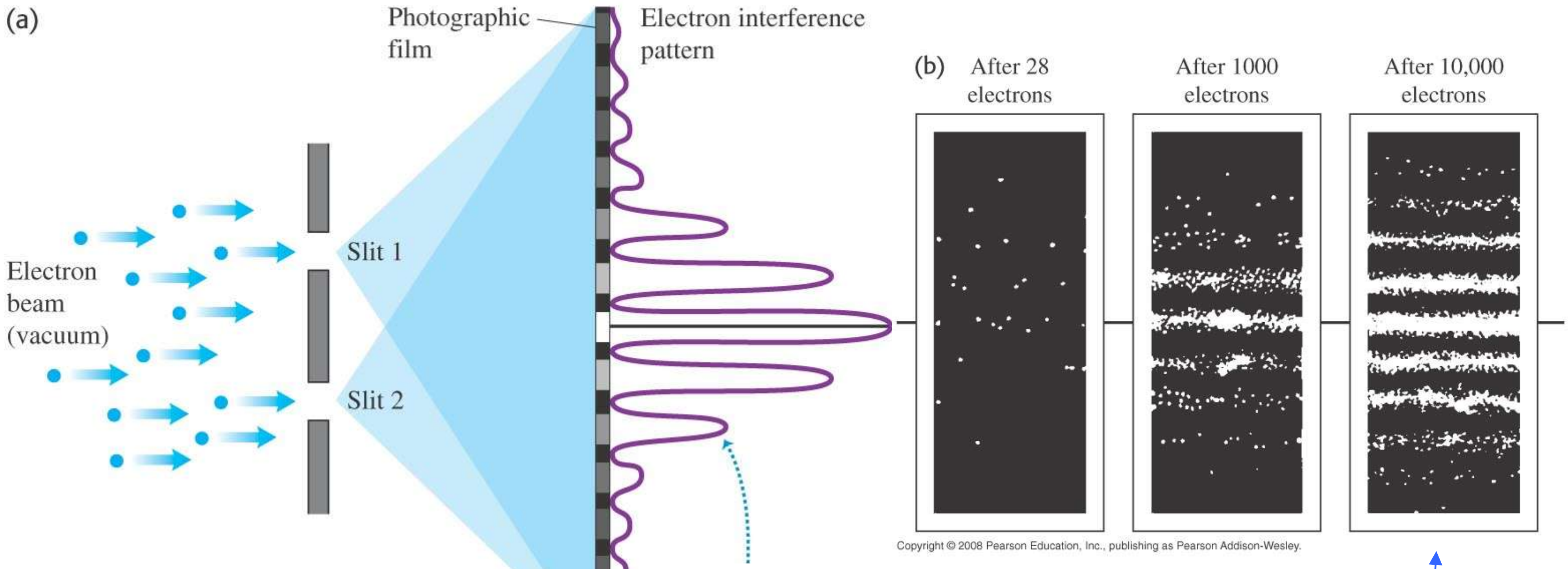
Top: x-ray diffraction



Bottom: electron diffraction

Electron Two-Slit Interference

Jonsson, Claus (Germany) Journal of Physics **161** (1961), p. 454-474 (first experiment)



The probability of arrival vs. *y* can only be explained by two *interfering* **matter waves** !

Similar to the two-slit experiment with photons, electrons demonstrate *both* particle (detection of single *e*'s on film) and wave (interference pattern) characteristics.

de Broglie wavelength: an electron vs. a baseball

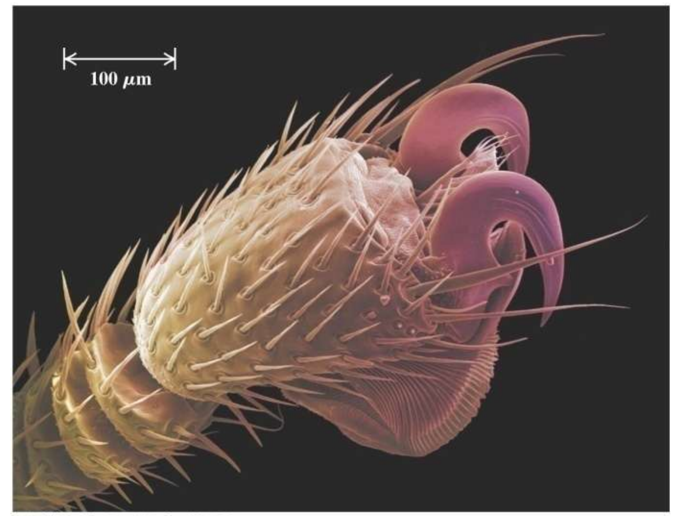
an electron moving at

$v = 1.00 \times 10^{7} m/s$

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} J \cdot s}{\left(9.11 \times 10^{-31} kg\right)\left(1.00 \times 10^{7} m/s\right)}$$

$$= 7.28 \times 10^{-11} m$$

(experimentally assessable)



a baseball moving at

$v = 40.0 m/s$

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} J \cdot s}{\left(145 \times 10^{-3} kg\right)\left(40.0 m/s\right)}$$

$$= 1.14 \times 10^{-34} m$$

(immeasurably small !)

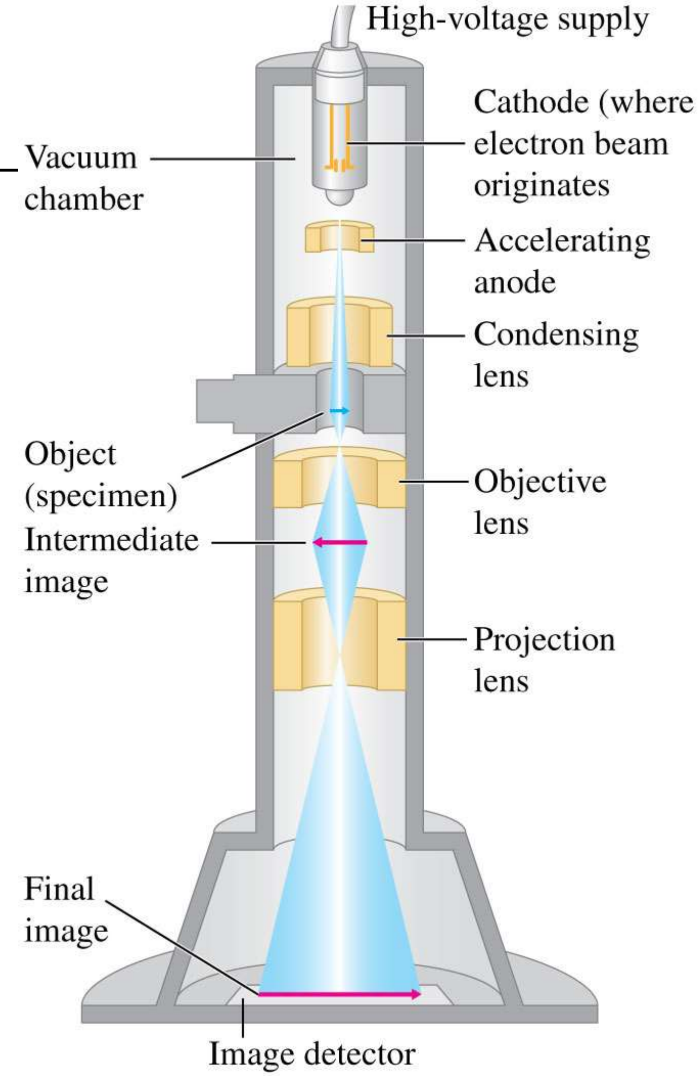
Since *h* is so small, typical daily object's wave characteristic is insignificant !

The Electron Microscope

- Similar to a light beam, a beam of electrons can be bent by reflection and refraction using electric and/or magnetic fields.

- Using electric/magnetic fields as "lens", an electron beam can be used as a microscope to form magnified images of an object.

- **Resolution of a microscope is limited by diffraction effects !**

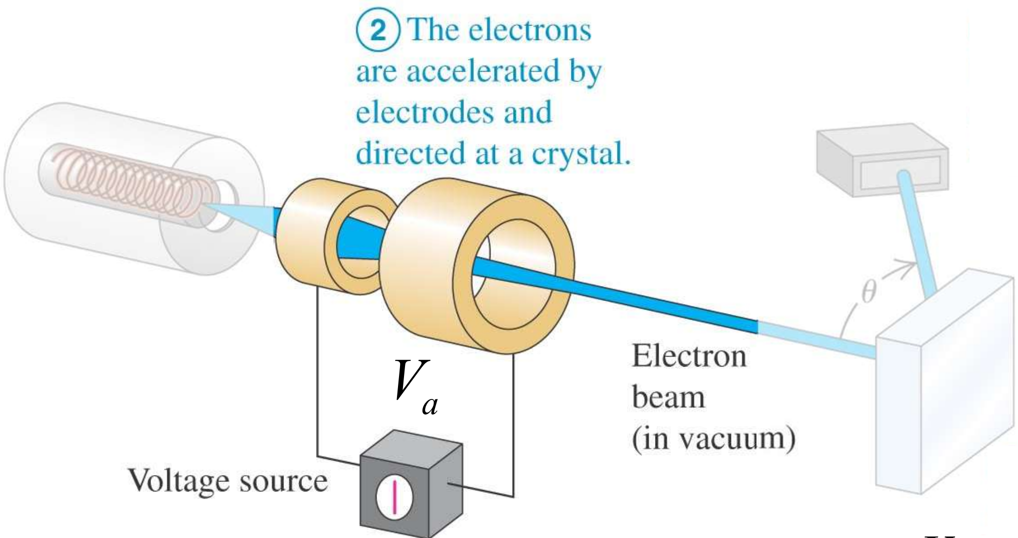


→ Electron microscope is better than regular microscope since $\lambda_{electron} \ll \lambda_{visible\ light}$

The Electron Microscope (Example 39.3)

What accelerating voltage is needed to produce e with $\lambda = 0.010 nm$?

From the electron diffraction experiment, we know how to relate the wavelength for a fast moving electron to the accelerating voltage V_a

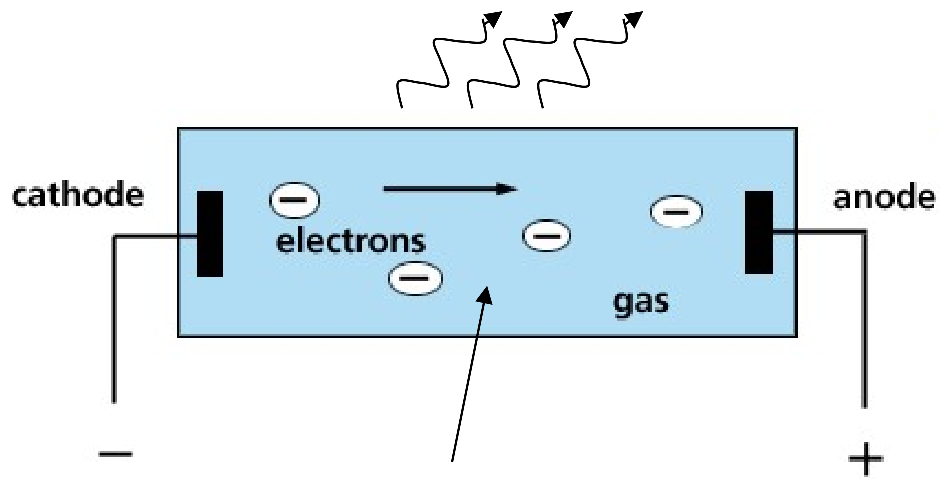


$$\left.\begin{aligned} eV_a &= p^2/2m \\ p &= h/\lambda \end{aligned}\right\} \quad V_a = \frac{h^2}{2me\lambda^2}$$

$$V_a = \frac{\left(6.626\times10^{-34} J\cdot s\right)^2}{2\left(9.109\times10^{-31} kg\right)\left(1.602\times10^{-19} C\right)\left(10\times10^{-12} m\right)^2}$$

$$= 1.5\times10^4 V = 15,000V$$

Atomic Spectra: Electric Discharge Tube with Diluted Gas



Diluted gas containing trace elements such as H, He, Na, Hg, …

Observation: Energetic electrons from cathode excite gaseous atoms in the tube, light can be emitted.

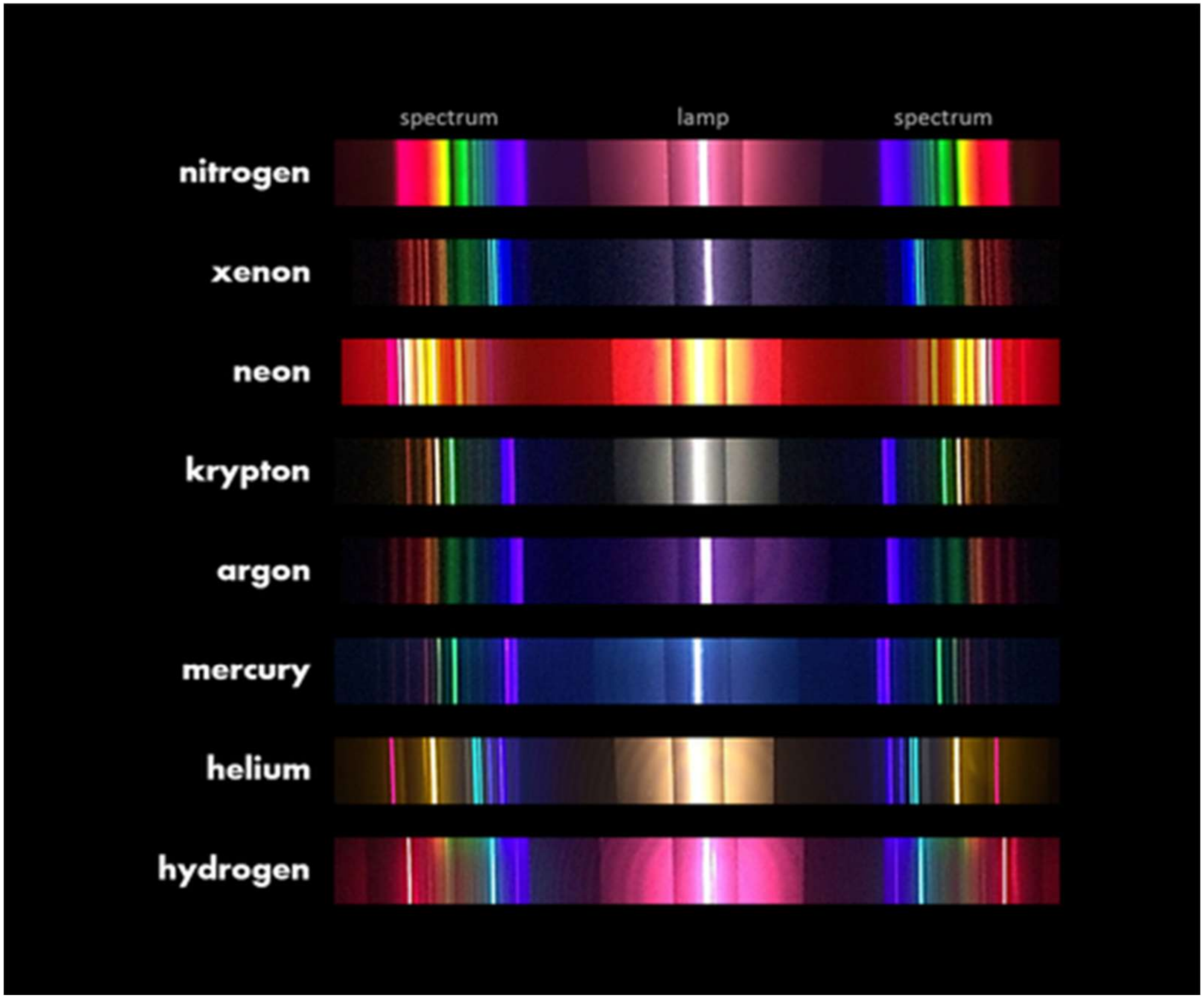
Gas is *diluted* so that the emission process is by *individual* atoms. The spectrum of light emitted are sets of unique lines characteristic of the specific type of atoms in the gas.

D discharge tube

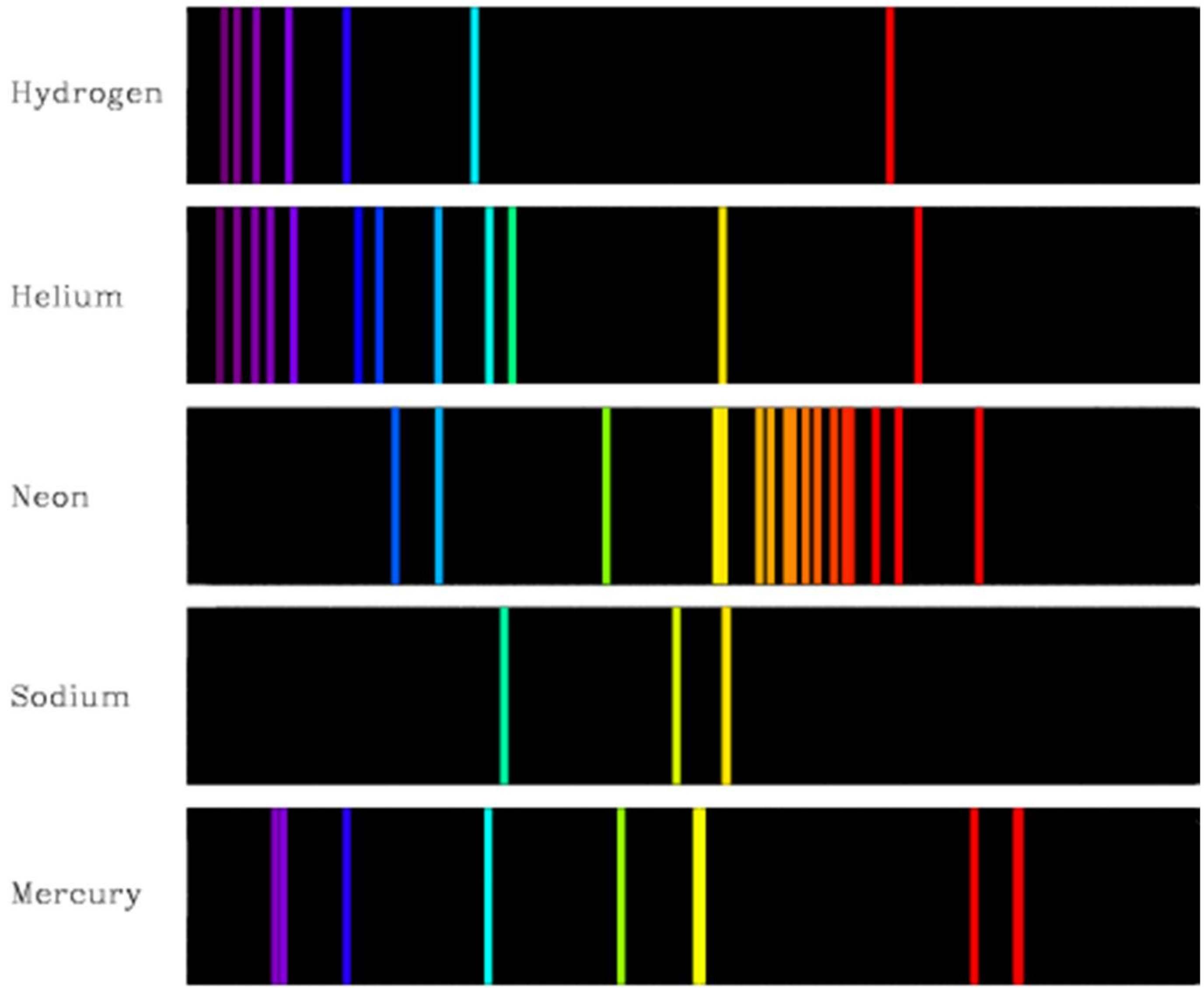
Emission Line Spectra

https://youtu.be/N_mwHxEugVE

Emission Line Spectra



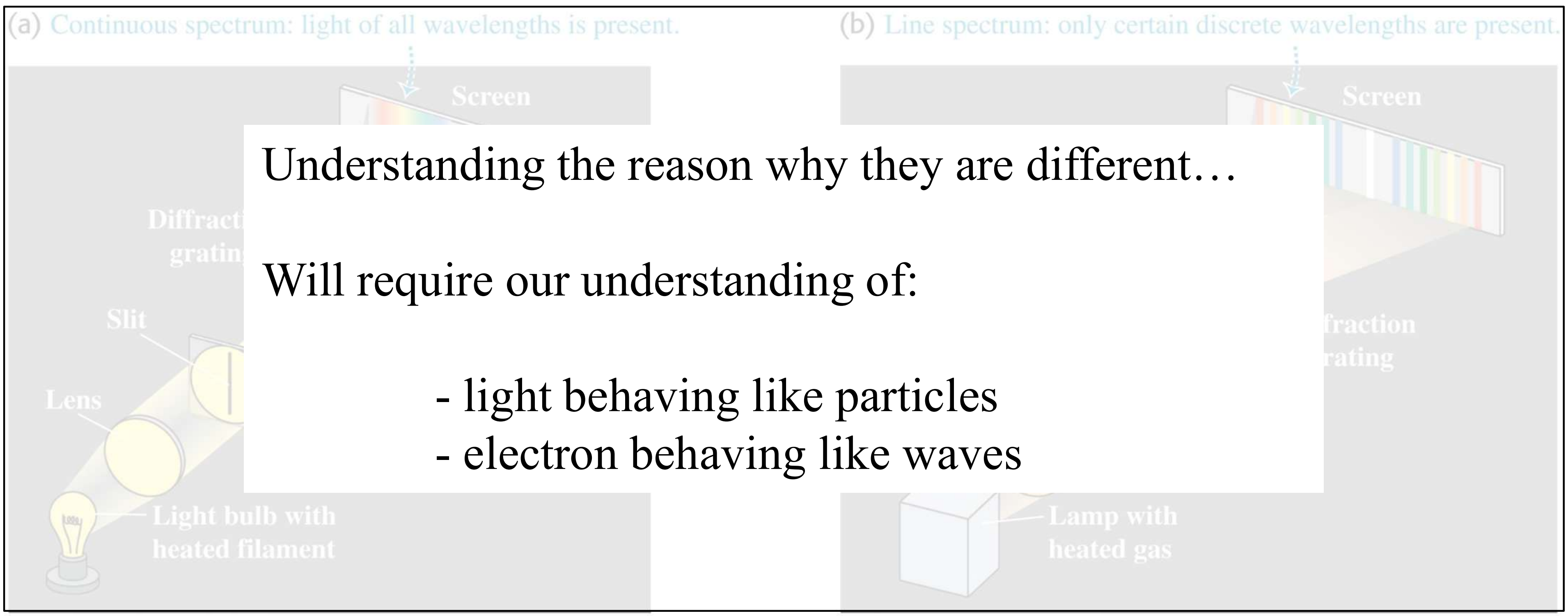
Atomic Spectra



Continuous vs. Line Spectra

When light emitted by a hot solid (e.g. filament in a light bulb) or liquid, the spectrum (from a diffraction grating or prism) is *continuous*

But light from the spectrum of a gas discharge tube is composed of *sharp lines*. Atoms/molecules in their gaseous state will give a set of unique wavelengths.



Understanding the reason why they are different…

Will require our understanding of:

- light behaving like particles
- electron behaving like waves